The Optimal Boundary and Regulator Design Problem for Event-Driven Controllers*

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Abstract. Event-driven control systems provide interesting benefits such as reducing resource utilization. This paper formulates the optimal boundary and regulator design problem that minimizes the resource utilization of an event-driven controller that achieves a cost equal to the case of periodic controllers.

1 Event-Driven Control System Model

We consider the control system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^{m\times 1}$, and $C \in \mathbb{R}^{1\times n}$. Let

$$u(t) = u_k = L\,x(a_k) = L\,x_k \qquad \forall t \in [a_k, a_{k+1}[\tag{2}$$

be the control updates given by a linear feedback controller designed in the continuous-time domain but using only samples of the state at discrete instants $a_0, a_1, \ldots, a_k, \ldots$ Between two consecutive control updates, $u(t)$ is held constant. In periodic sampling we have $a_{k+1} = a_k + h$, where h is the period of the controller.

Let $e_k(t) = x(t) - x_k$ be the error evolution between consecutive samples with $t \in [a_k, a_{k+1}[$. For several types of event-driven control approaches [1, 2], event conditions can be generalized by introducing a function $f(\cdot, \cdot, \Upsilon) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that defines a boundary measuring the tolerated error with to respect the sampled state [3]. The condition that must be ensured is

$$f(e_k(t), x_k, \Upsilon) \leq \eta \tag{3}$$

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where η is the error tolerance and $\Upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_p\}$, $\upsilon_i \in \mathbb{R}$ is a set of free parameters. Hence, we can define the complete dynamics of the event-driven system by the $n+1$ order non linear discrete-time system

$$\begin{aligned} a_{k+1} &= a_k + \Lambda(x_k, \Upsilon, \eta) \\ x_{k+1} &= (\Phi(\Lambda(x_k, \Upsilon, \eta)) + \Gamma(\Lambda(x_k, \Upsilon, \eta))L)x_k \end{aligned} \tag{4}$$

where $\Lambda(x_k, \Upsilon, \eta)$ denotes the time separation between two consecutive activations a_{k+1} and a_k, that solves (1), (2), and (3), assuming that $x_k = x(a_k)$ is the state sampled at a_k, Υ is the set of free parameters of f, and η is the tolerance to the error. We also define $\Phi(t) = e^{At}$ and $\Gamma(t) = \int_0^t e^{As}\mathrm{d}sB$. We highlight that we have been able to find an expression for $\Lambda(x_k, \Upsilon, \eta)$ only by approximating Φ and Γ by Taylor expansion [3]. In all the other cases Λ can only be computed numerically.

2 Optimal Problem Formulation

The optimal problem for event-driven controllers can be formulated in two complementary ways: to minimize the cost while using the same amount of resources than the periodic controller, or to minimize the computational demand while achieving the same cost as in the case of the periodic controller. Here we describe the resource usage minimization given a cost constraint. The other formulation simply requires to exchange the goal function and one constraint, as it will be indicated later.

Let be a standard quadratic cost function in continuous time defined as

$$J(L, \Upsilon, \eta) = \int_0^{a_\ell} x(t)^T Q_c x(t) + u(t)^T R_c u(t)\mathrm{d}t + x(a_\ell)^T N_c x(a_\ell) \tag{5}$$

The optimal boundary and regulator design problem for resource minimization can be formulated as

$$\text{maximize} \quad \frac{\sum_{k=0}^{\ell-1} \Lambda(x_k, \Upsilon, \eta)}{k} \qquad \text{w.r.t. } L, \Upsilon, \eta \tag{6}$$

$$\text{subject to} \quad x_{k+1} = (\Phi(\Lambda(x_k, \Upsilon, \eta)) + \Gamma(\Lambda(x_k, \Upsilon, \eta))L)x_k \tag{7}$$

$$a_{k+1} = a_k + \Lambda(x_k, \Upsilon, \eta) \tag{8}$$

$$J(L, \Upsilon, \eta) \leq J_h \tag{9}$$

where (6) sets the maximization goal equal to the average of the first ℓ sampling intervals, (7) enforces the relationship between two consecutive sampled states, (8) describes the constraint among the activations, and J_h is the cost of an optimal h-periodic controller.

Notice that by exchanging (6) with (9) we obtain the complementary problem that minimizes the cost given an upper bound on the period.

The problem (6)–(9) can be numerically solved by constrained minimization techniques such as Lagrange multipliers, or by standards procedures for time varying discrete-time systems.

3 Example

Consider the double integrator system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

A first closed loop system using a periodic optimal regulator designed using standard methods to minimize (5) with $h = 0.6$s gives a cost of 27.3648 with $L = \begin{bmatrix} -0.6115 & -1.2637 \end{bmatrix}$, where

$$x_0 = \begin{bmatrix} 0.54 \\ 0.84 \end{bmatrix}, Q_c = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, R_c = \begin{bmatrix} 10 \end{bmatrix}, N_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, a_n = 100s.$$

Alternatively, for an event-driven controller, let

$$\dot{x}_{k^+}^T M_1 \dot{x}_{k^+} (a_{k+1} - a_k)^2 = \eta x_k^T M_2 x_k \tag{10}$$

be an execution rule as in (3) that intuitively mandates to trigger more frequent control updates when the state moves fast. In (10) we set

$$\dot{x}_{k^+} = \lim_{t \to a_k^+} \dot{x}(t) = (A + BL) x_k \tag{11}$$

to denote the state derivative once the controller has been applied to the sampled state. From (10), it follows that

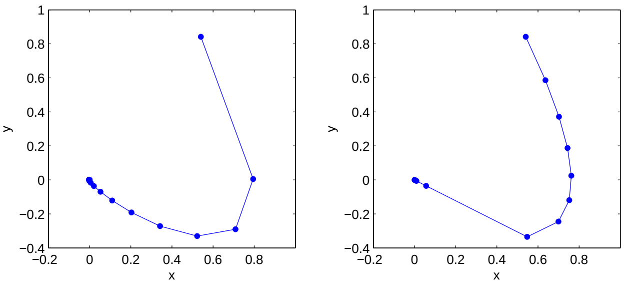
$$a_{k+1} - a_k = \Lambda(x_k, \Upsilon, \eta) = \sqrt{\eta \frac{x_k^T M_2 x_k}{x_k^T (A+BL)^T M_1 (A+BL) x_k}}. \tag{12}$$

The optimal problem (6)–(9) is completely defined except for (9). Note that for each optimization problem, 9 free parameters have been defined (6 for both positive semidefinite $M_{1,2}$, 1 for η, and 2 for L). Considering for example problem (6)–(9), the optimal solution achieves a slightly better cost than the optimal periodic controller, 26.6005, with

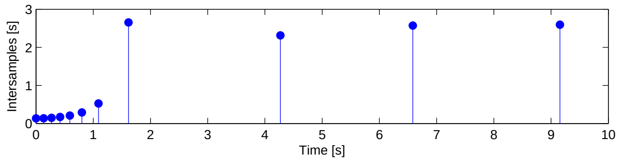
$$L = \begin{bmatrix} -0.847 & -1.723 \end{bmatrix}, M_1 = \begin{bmatrix} 0.028 & 0.091 \\ 0.091 & 0.336 \end{bmatrix}, M_2 = \begin{bmatrix} 0.054 & 0.017 \\ 0.017 & 0.069 \end{bmatrix}, \eta = 0.0.212,$$

but drastically reducing resource utilization.

Figure 1 a) shows the closed loops dynamics of the periodic optimal controller and event-driven controller respectively, where circles mark control updates. Both trajectories exhibit similar dynamics. Focusing on the dynamics given by the periodic controller, we can observe that from the first to the second control update, the state moves fast because it covers a long trajectory. And as control updates progress, the covered trajectories become shorter (the state moves slow). Looking at the dynamics given by the event-driven controller, we can observe the opposite behavior. When the state moves fast, we have more frequent control updates than when the state moves slow.



(a) Discrete and event-driven closed loop dynamics



(b) Event-driven controller activation times

Fig. 1: Numerical example.

Figure 1 b) shows the activation pattern of control updates for the event-driven controller. The x-axis is simulation time, and each control update is represented by a vertical line, whose height indicates the time (in seconds) elapsed to the next control update. It shows that activation times occur within a range ([0.1 2.6]s, approximately) that in average is 1.84s, three times slower than the periodic controller!! Only the first 10s of simulation time are shown in this subfigure. By looking at the rest of simulation time, we would observe that sampling intervals oscillate within 2.51s and 2.58s.

4 Conclusions

This paper has formalized two optimal control design problems for event-driven controllers with limited resource utilization. The formalization includes a restriction on the amount of resources to be spent or on the cost to be achieved. Future work will look for closed solutions to the problem.

References

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