



Wealth-driven selection in a financial market with heterogeneous agents[☆]

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ABSTRACT

We study the co-evolution of asset prices and individual wealth in a financial market with an arbitrary number of heterogeneous boundedly rational investors. Using wealth dynamics as a selection device we are able to characterize the long run market outcomes, i.e., asset returns and wealth distributions, for a general class of competing investment behaviors. Our investigation illustrates that market interaction and wealth dynamics pose certain limits on the outcome of agents' interactions even within the "wilderness of bounded rationality". As an application we consider the case of heterogeneous mean-variance optimizers and provide insights into the results of the simulation model introduced by Levy, Levy and Solomon (1994).

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1. Introduction

Consider a financial market where a group of heterogeneous investors, each following a different strategy to gain superior returns, is trading. The open questions are to specify how their interaction affects market returns and who will survive in the long run. This paper seeks to give a contribution to this issue by investigating the co-evolution of asset prices and agents' wealth in a stylized market for a long-lived financial asset with an arbitrary number of heterogeneous agents. We do so under three main assumptions. First, asset demands are proportional to agents' wealth, so that market clearing prices and agents' wealth co-evolve. Second, each individual investment behavior can be formalized as a function of past returns. Third, the dividends of the risky asset follow a geometric random walk.

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By focusing on asset prices dynamics in a market with heterogeneous investors, our paper clearly belongs to the growing field of Heterogeneous Agent Models (HAMs), see [Hommes \(2006\)](#) for a recent survey. We share the standard set-up of this literature and assume that agents decide whether to invest in a risk-free bond or in a risky financial asset.² In the spirit of [Brock and Hommes \(1997\)](#) and [Grandmont \(1998\)](#) we consider a stochastic dynamical system and analyze the sequence of temporary equilibria of its deterministic skeleton.

Whereas the majority of HAMs consider only a few types of investors and concentrate on heterogeneity in expectations, our framework can be applied to a quite large set of investment strategies so that heterogeneity with respect to risk attitude, expectations, memory and optimization task can be accommodated. Employing the tools developed in [Anufriev and Bottazzi \(2009\)](#) we are able to characterize the long-run behavior of asset prices and agents' wealth for a general set of competing investment strategies, which can be specified as a function of past returns.

An important feature of our model concerns the demand specification. In contrast to many HAMs (see, e.g., [Brock and Hommes, 1998](#); [Gauersdorfer, 2000](#); [Brock et al., 2005](#)), which employ the setting where agents' demand exhibits constant absolute risk aversion (CARA), we assume that demand increases linearly with agents' wealth; that is, it exhibits constant relative risk aversion (CRRA). In such a setting agents affect market price proportionally to their relative wealth. As a consequence, relative wealth represents a natural measure of performance of different investment behaviors. On the contrary, in CARA models the wealth dynamics does not affect agents' demand, implying that a performance measure has to be introduced *ad hoc* time by time. Furthermore, experimental literature seems to lean in favor of CRRA rather than CARA (see, e.g., [Kroll et al., 1988](#) and Chapter 3 in [Levy et al., 2000](#)).

The analytical exploration of the CRRA framework with heterogeneous agents is difficult because the wealth dynamics of every agent has to be taken into account. Although there has been some progress in the literature (see, e.g., [Chiarella and He, 2001, 2002](#); [Anufriev et al., 2006](#); [Anufriev, 2008](#) and [Anufriev and Bottazzi, 2009](#)), all these studies are based on the assumption that the price-dividend ratio is exogenous. This seems at odd with the standard approach, where the dividend process is exogenously set, while the asset prices are endogenously determined. In our paper, to overcome this problem, we analyze a market for a financial asset whose dividend process is exogenous, so that the price-dividend ratio is a dynamic variable. Our paper can thus be seen as an extension of [Anufriev and Bottazzi \(2009\)](#) to the case of exogenous dividends.³

As a result we show that depending on the difference between the growth rate of dividends and the risk-free rate, which are the exogenous parameters of our model, the dynamics can converge to two types of equilibrium steady-states. When the growth rate of dividend is higher than the risk-free rate, the equilibrium dividend yield is positive, asset gives a higher expected return than the risk-free bond, and only one or few investors have a positive wealth share. Only such "survivors" affect the price in a given steady-state. However, multiple steady-states with different survivors and different levels of the dividend yield are possible, and the range of possibilities depends on the whole ecology of traders. Otherwise, when the dividends' growth rate is smaller than the risk-free rate, the dividend yield goes to zero, both the risky asset and the risk-free bond give the same expected return, and the wealth of all agents grows at the same rate as asset prices. Notice, however, that convergence to either types of steady-state equilibria is not granted. We show how local stability of each steady-state depends on the strength of the price feedback, occurring via the investment functions.

An important reason for departing from previous works with CRRA demands is that it allows for a direct application to a well known simulation model. In fact, our CRRA setup with exogenous dividend process is identical to the setup of one of the first agent-based simulation model of a financial market introduced by [Levy, Levy and Solomon \(LLS model, henceforth\)](#); see, e.g., [Levy et al. \(1994\)](#). Their work investigates whether stylized empirical findings in finance, such as excess volatility or long periods of asset overvaluation, can be explained by relaxing the assumption of a fully-informed, rational representative agent. Despite some success of the LLS model in reproducing the financial "stylized facts", all its results are based on simulations. Our general setup can be applied to the specific demand schedules used in the LLS model and, thus, provides an analytical support to its simulations.

As we are looking at agents' survival in a financial market ecology, our work can be also classified within the realm of *evolutionary finance*. The seminal work of [Blume and Easley \(1992\)](#), as well as more recent papers of [Sandroni \(2000\)](#); [Hens and Schenk-Hoppé \(2005\)](#); [Blume and Easley \(2006, 2009\)](#) and [Evstigneev et al. \(2006\)](#), investigate how beliefs about the dividend process affect agents' dominance in the market. A key difference between our model and the evolutionary finance approach is that our agents can condition their investment decisions on past values of endogenous variables such as prices. As a consequence, in our framework prices today influence prices tomorrow through their impact on agents' demands, generating a price feedback mechanism. In the HAMs such mechanism plays an important role for the stability of dynamics. For instance, when the investment strategy is too responsive to price movements, fluctuations are typically amplified and unstable price dynamics are produced. Indeed, we show that local stability is related to how far agents look in the past.

This paper is organized as follows. Section 2 presents the model and leads to the definition of the stochastic dynamical system where prices and wealths co-evolve. The steady-states of the deterministic version of the system are studied in

² Recently, also some models with heterogeneous agents operating in markets with multiple assets ([Chiarella et al., 2007](#)) and with derivatives ([Brock et al., 2006](#)) have been developed.

³ The CRRA framework with exogenously growing dividends has been also investigated in [Chiarella et al. \(2006a\)](#), but under a different mechanism of market clearing, i.e., market-maker scenario. The focus of their analysis is also somewhat different from ours. They study the return dynamics with two specific types of traders, fundamentalists and chartists, rather than in general as we do.

Section 3 when only one or two investors are trading. At this level of the analysis the investment behavior is left unspecified, but the process of wealth accumulation enable us to characterize the locus of possible steady-states and specify conditions for their local stability. Section 4 applies the former analysis to the special case where agents are mean-variance optimizers. Section 5 extends the analysis of Section 3 to the general case of N investors. Section 6 uses these last analytical results to explain the simulations of the LLS model. Section 7 summarizes our main findings and concludes. Most proofs are collected in Appendices at the end of the paper.

2. The model

Let us consider a group of N agents trading in discrete time in a market for a long-lived risky asset. Assume that the asset is in constant supply which, without loss of generality, can be normalized to 1. Alternatively, agents can buy a riskless asset whose return is constant and equal to $r_f > 0$. The riskless asset serves as numéraire with price normalized to 1 in every period. At time t the risky asset pays a dividend D_t in units of the numéraire, while its price P_t is fixed through market clearing.

Let $W_{n,t}$ stand for the wealth of agent n at time t . It is convenient to express agents' demand for the risky asset in terms of the fraction $x_{n,t}$ of wealth invested in this asset, so that the amount of the risky asset bought at time t by agent n is $x_{n,t}W_{n,t}/P_t$. The dividend is paid before trade takes place, and the wealth of agent n evolves according to

$$W_{n,t+1} = (1 - x_{n,t})W_{n,t}(1 + r_f) + x_{n,t}W_{n,t} \frac{P_{t+1} + D_{t+1}}{P_t}. \tag{2.1}$$

The asset price at time $t + 1$ is fixed through the market clearing condition

$$\sum_{n=1}^N x_{n,t+1} \frac{W_{n,t+1}}{P_{t+1}} = 1. \tag{2.2}$$

Prices and wealth co-evolve because prices depend on current agents' wealth via (2.2) and, at the same time, the wealth of every agent depends on the contemporaneous price via (2.1). In what follows, first, we solve these equations and make the dynamics of prices and wealth explicit, assuming that dividend process and individual investment decisions are given. Then, we give a precise specification of both investment decisions and dividend process. Finally, we summarize our model as a dynamical system and introduce the notion of agents' survival.

2.1. Co-evolution of wealth and prices

In order to obtain an explicit solution for P_{t+1} , and thereafter W_{t+1} , one can use (2.1) to rewrite the market clearing equation (2.2) as

$$P_{t+1} = \sum_{n=1}^N x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \left(\frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} \right) \right).$$

The solution of this equation with respect to P_{t+1} gives

$$P_{t+1} = \frac{\sum_{n=1}^N x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \frac{D_{t+1}}{P_t} \right)}{1 - \frac{1}{P_t} \sum_{n=1}^N W_{n,t} x_{n,t+1} x_{n,t}}, \tag{2.3}$$

which using (2.1) fixes the level of individual wealth $W_{n,t+1}$ for every agent n . The resulting expressions can be conveniently written in terms of price returns, dividend yields, and each agent's relative wealth,³ defined respectively as

$$k_{t+1} = \frac{P_{t+1}}{P_t} - 1, \quad y_{t+1} = \frac{D_{t+1}}{P_t} \quad \text{and} \quad \varphi_{n,t} = \frac{W_{n,t}}{\sum_m W_{m,t}}.$$

Dividing both sides of (2.3) by P_t and using that $P_t = \sum x_{n,t}W_{n,t}$, one can rewrite the dynamics in terms of price returns. This, together with the resulting expression for the evolution of wealth shares, gives the following system

$$\begin{cases} k_{t+1} = r_f + \frac{\sum_n ((1 + r_f)(x_{n,t+1} - x_{n,t}) + y_{t+1} x_{n,t} x_{n,t+1}) \varphi_{n,t}}{\sum_n x_{n,t} (1 - x_{n,t+1}) \varphi_{n,t}}, \\ \varphi_{n,t+1} = \varphi_{n,t} \frac{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f) x_{n,t}}{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f) \sum_m x_{m,t} \varphi_{m,t}}, \quad \forall n \in \{1, \dots, N\}. \end{cases} \tag{2.4}$$

³ Notice that this change of variables changes the nature of steady-states equilibria, from constant levels to constant changes. The latter is more appropriate in an economy which is possibly growing, like ours. For the same reason, other works in the literature, such as Chiarella and He (2001) and Anufriev et al. (2006), take the same approach.

According to the first equation returns depend on agents' investment decisions for two consequent periods. High investment fractions for the current period tend to increase current prices and, hence returns, due to an increase of current demand. The overall effect of agents' decisions on price returns is proportional to their relative wealth. The second equation shows that each agent relative wealth changes according to his *relative* performance, as given by his portfolio returns.

2.2. Investment functions

We intend to study the evolution of asset prices and agents' wealth while keeping the investment strategies as general as possible. Therefore we avoid any explicit formulation of the demand and suppose that the fraction of wealth invested in the risky asset, $x_{n,t}$, are general functions of past realizations of price returns and dividend yields. Following Anufriev and Bottazzi (2009) we formalize this concept of investment strategy as follows.

Assumption 2.1. For each agent $n = 1, \dots, N$ there exists an *investment function* f_n which maps the information set into an investment share:

$$x_{n,t} = f_n(k_{t-1}, k_{t-2}, \dots, k_{t-L}; y_t, y_{t-1}, \dots, y_{t-L}). \quad (2.5)$$

Agents' investment decisions evolve following individual prescriptions and depend in a general way on the available information set.⁴ The investment choices of period t should be made before trade starts, i.e., when the price P_t is still unknown. Thus, the information set contains past price returns up to k_{t-1} and past dividend yields up to y_t .

Assumption 2.1 leaves a high freedom in the demand specification. The only essential restrictions are stationarity, i.e. the same information observed in different periods is mapped to the same investment decision, and that the investment share does not depend on the contemporaneous wealth. This implies that the demand of trader i is linearly increasing with its own wealth. In other words, *ceteris paribus* investors maintain a constant proportion of wealth invested in the risky asset as their wealth level changes. Such behavior can be referred to as constant relative risk aversion (CRRA).⁵

A number of standard demand specifications are consistent with Assumption 2.1. In Section 4 we consider agents who maximize mean-variance utility of (next period) expected return. Alternatively, one can consider agents behaving in accordance with the prospect theory of Kahneman and Tversky (1979).⁶ The generality of our investment functions allows modeling *forecasting* behavior with a big flexibility too. Formulation (2.5) includes as special cases both technical trading, e.g., when investment decisions are driven by the observed price fluctuations, and more fundamental attitudes, e.g., when the decisions are made on the basis of the price-dividend ratio. It also includes the case of a constant investment strategy, which corresponds to agents assuming a stationary *ex-ante* return distribution.

Despite its high generality, our setup does not include a number of important investment behaviors. Since current wealth is not included as an argument of the investment function, all the demand functions of CARA type cannot be accomplished. Also the current price is not among the arguments of (2.5). Therefore, conditioning the investment share on the current price cannot be reconciled with our setup.

Finally notice that some investment functions of the type (2.5) may lead to a dynamics which is not economically sensible, e.g., with negative prices. As it is shown in Proposition 5.1, to avoid such instances it is sufficient to forbid investors to take short positions. For this reason, we complement Assumption 2.1 with the following

Assumption 1'. For each $n = 1, \dots, N$ the investment function f_n is restricted to assume values in the open interval $(0, 1)$.

2.3. Dividend process

The last ingredient of the model is the dividend process. The previous analytical models built in a similar framework, such as Chiarella and He (2001); Anufriev et al. (2006); Anufriev (2008) and Anufriev and Bottazzi (2009), assume that the dividend yield is an i.i.d. process. This implies that any change in the level of prices causes an immediate change in the level of dividends. In reality, however, the dividend policy of firms is hardly so fast responsive to the performance of firms' assets, especially when prices do not reflect fundamentals, e.g., in a speculative bubbles. Both for the sake of reality and for comparison to previous works on the LLS model, we find it interesting to investigate what happens when the dividend process is an exogenous process. For these reasons, this is the approach we take in this paper.

Assumption 2.2. The dividend realization follows a geometric random walk,

$$D_t = D_{t-1} (1 + g_t),$$

where the growth rate, g_t , is an i.i.d. random variable.

⁴ In order to deal with a finite dimensional dynamical system, we restrict the memory span of each agent to a finite L . Notice however that L can be arbitrarily large.

⁵ The distinction between constant relative and constant absolute risk aversion (CARA) behavior was introduced in Arrow (1965) and Pratt (1964), who also relates these concepts with utility maximization. Under CARA framework agents maintain a constant demand for the risky asset as their wealth changes.

⁶ This is shown, for instance, in Chapter 9 of Levy et al. (2000).

Rewriting this assumption in terms of dividend yields and price returns we get

$$y_{t+1} = y_t \frac{1 + g_{t+1}}{1 + k_t}. \tag{2.6}$$

2.4. Dynamics, steady-state equilibria and agents' survival

Equations (2.4), (2.5) for each $n = 1, \dots, N$, and (2.6) specify the evolution of the asset market with N heterogeneous agents. The dynamics of the model is stochastic due to the fluctuations of the dividend process. Following the typical route in the literature (cf. Brock and Hommes, 1997; Grandmont, 1998), our analysis concentrates on the *deterministic skeleton* of this dynamics, obtained by fixing⁷ the growth rate of dividends g_t at the constant level g , thus leading to the following dynamical system⁸

$$\begin{cases} x_{n,t+1} = f_n(k_t, k_{t-1}, \dots, k_{t-L+1}; y_{t+1}, y_t, \dots, y_{t-L+1}), & \forall n \in \{1, \dots, N\} \\ \varphi_{n,t+1} = \varphi_{n,t} \frac{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)x_{n,t}}{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f) \sum_m x_{m,t} \varphi_{m,t}}, & \forall n \in \{1, \dots, N\}, \\ k_{t+1} = r_f + \frac{\sum_n ((1 + r_f)(x_{n,t+1} - x_{n,t}) + y_{t+1} x_{n,t} x_{n,t+1}) \varphi_{n,t}}{\sum_n x_{n,t} (1 - x_{n,t+1}) \varphi_{n,t}}, \\ y_{t+1} = y_t \frac{1 + g}{1 + k_t}. \end{cases} \tag{2.7}$$

In what follows we are primarily concerned with the steady-states of this system and their local stability. In a steady-state aggregate economic variables, such as price returns and dividend yields, are constant and can be denoted by k^* and y^* , respectively. Every steady-state has also constant investment shares (x_1^*, \dots, x_n^*) and wealth distribution $(\varphi_1^*, \dots, \varphi_n^*)$.

Given the arbitrariness of the population size N , of the memory span L , and the absence of any specification for the investment functions, the analysis of the dynamics generated by (2.7) is non-trivial in its general formulation. However, as we show in the rest of the paper, the constraints on the dynamics set by the dividend process, the market clearing, and the wealth evolution are sufficient to (i) uniquely characterize the steady-state level of price returns k^* , (ii) restrict the possible values of steady-state dividend yields y^* to a low-dimensional set, (iii) describe the corresponding distributions of wealth among agents. Moreover it is possible to derive general conditions under which convergence to these equilibria is guaranteed.

When characterizing the wealth distribution, the following terminology is useful

Definition 2.1. An agent n is said to *survive on a given trajectory* generated by dynamics (2.7) if $\limsup_{t \rightarrow \infty} \varphi_{n,t} > 0$ on this trajectory. Otherwise, an agent n is said to *vanish on a trajectory*.

Notice that we have defined survivability only with respect to a given trajectory and not in general. The reason is that a trader may survive on one trajectory (i.e., for certain initial conditions) but vanish on another. A similar definition is given in Blume and Easley (1992), for a stochastic system, and in Anufriev and Bottazzi (2009).

Applying the previous definition to a steady-state, which is the simplest trajectory, we shall say that an agent n survives at the steady-state equilibrium $(x_1^*, \dots, x_n^*; \varphi_1^*, \dots, \varphi_n^*; k^*; y^*)$ if his wealth share is strictly positive, $\varphi_n^* > 0$, while vanishes if $\varphi_n^* = 0$. Such taxonomy can be equally applied to a stable and an unstable steady-state, but the implications are very different in these two cases. When the steady-state is stable, all trajectories starting in a neighborhood of it converge to this steady-state, and a survivor at the steady-state also survive on all these trajectories (with the wealth share converging to the equilibrium value). On the other hand, trajectories started close to an unstable steady-state may behave very differently from the steady-state itself. Both vanishers and survivors at an unstable steady-state can survive as well as vanish on trajectories started in its neighborhood.

The equilibrium and stability analysis of the deterministic skeleton gives a considerable insight into the stochastic dynamics performed when g_t is a random variable. In fact, as long as the stochastic fluctuations are small enough, survival at a stable steady-state carry over also for typical trajectories of the stochastic system started at the steady-state. Survival at an unstable steady-state, instead, says nothing about agent's wealth share in the corresponding stochastic dynamics. See Section 4.1 for examples.

3. Market dynamics with few agents

In this and the next sections we consider a market where only one or two agents are trading. The purpose is to get an overview of the different price and wealth dynamics that the model can generate and give an insight into their underlying mechanisms. As the results are special cases of the general model with N traders analyzed in Section 5, we skip all the proofs.

⁷ It is important to keep in mind that agents do take the risk due to randomness of dividends into account, when deriving their investment functions. Given agents' behavior, we, as the modelers, set the noise level to zero and analyze the resulting deterministic dynamics.

⁸ See Appendix A for an explicit one-step operator associated with this system.

The main message of this section is that the concept of Equilibrium Market Curve (EMC) introduced in Anufriev et al. (2006) and Anufriev and Bottazzi (2009) enables us to tell quite a few things regarding the location of steady-state equilibria and their stability, even with unspecified investment functions.

3.1. Single agent

Since the relative wealth dynamics can be ignored, markets with a single investor are the easiest to analyze. Omitting agent-specific subindices, the dynamics (2.7) can be written as

$$\begin{cases} x_{t+1} = f(k_t, k_{t-1}, \dots, k_{t-L+1}; y_{t+1}, y_t, \dots, y_{t-L+1}), \\ k_{t+1} = r_f + \frac{(1+r_f)(x_{t+1} - x_t) + y_{t+1} x_t x_{t+1}}{x_t(1-x_{t+1})}, \\ y_{t+1} = y_t \frac{1+g}{1+k_t}. \end{cases} \quad (3.1)$$

The steady-states equilibria of (3.1) correspond to constant values of the investment share x^* , return k^* and dividend yield y^* . The last equation suggests that two cases should be distinguished, when y^* is positive and when y^* is zero.

3.1.1. Location of steady-states

Positive yield. Assume, first, that the equilibrium dividend yield is positive. To sustain a constant yield prices should grow at the same rate as the dividend, i.e., from the last equation of (3.1), $k^* = g$. Substitution into the first two equations leads to

$$x^* = f(\underbrace{g, \dots, g}_L; \underbrace{y^*, \dots, y^*}_{L+1}) \quad \text{and} \quad x^* = \frac{g - r_f}{y^* + g - r_f}, \quad (3.2)$$

which, g and r_f being exogenous, represents a system of two equations in two variables, the investment share x^* and the dividend yield y^* . When the investment function is specified any solution of this system, satisfying the restrictions $y^* > 0$ and, due to Assumption 2.2, $x^* \in (0, 1)$, completes the computation of the steady-states.

Even if the investment function is left unspecified, we are able to characterize the set of possible steady-state yields and investment shares. First of all, notice that for $g \leq r_f$ no solution with $y^* > 0$ and $x^* \in (0, 1)$ exists. Thus, we assume that $g > r_f$ and look for solutions in this case. Since both equations in (3.2) define a one-dimensional curve in coordinates (y, x) , the problem can be solved graphically using the intersections of the plots of two functions. The first function, which can be referred to as the Equilibrium Investment Function (EIF), is a cross-section of the investment function f by the set

$$\{k_t = k_{t-1} = \dots = k_{t-L+1} = g; \quad y_{t+1} = y_t = \dots = y_{t-L+1} = y\},$$

We use the tilde sign to distinguish the single-variable EIF from an original multi-variable investment function, i.e.,

$$\tilde{f}(y) = f(\underbrace{g, \dots, g}_L; \underbrace{y, \dots, y}_{L+1}). \quad (3.3)$$

For any value of the dividend yield, the EIF gives that agent's investment share which is consistent with the condition of constant yield (i.e., with prices growing at rate g). The second function, whose plot is called the Equilibrium Market Curve (EMC),⁹ is defined as

$$l(y) = \frac{g - r_f}{y + g - r_f} \quad \text{for } y > 0. \quad (3.4)$$

For any y , it determines the investment share necessary for having prices growing at rate g . At the steady-state, where (3.2) holds and the plots of the functions $\tilde{f}(y)$ and $l(y)$ intersect, the agent invests as much as needed to generate the constant yield economy (since the point belongs both to the EIF and the EMC).

In the left panel of Fig. 1 the EMC is drawn as a thick curve, and one example of the EIF is shown as a thin curve. The values of yields and investment shares in all possible steady-states of (3.1) can be found as the coordinates of the intersections of the EMC with the EIF, in this case of the points A and B. To compare these two steady-states, notice that the EMC is a decreasing function. Thus, the steady-states with more "aggressive" behavior (i.e., investment of larger wealth fraction in the risky asset) deliver a smaller equilibrium yield. In point A the agent invests a high proportion of his wealth in the risky asset. Such investment behavior pushes the price up and leads to a low yield. In point B the investment share is lower, leading to a higher yield. More aggressive behavior pushes the demand up and leads to a higher price level, which decreases the

⁹ The name EMC was introduced in Anufriev et al. (2006) and Anufriev and Bottazzi (2009). In these papers, due to the different assumptions, the EMC defines a curve in the coordinates "price return – investment share" and has a different functional form than (3.4). Since in both our and these previous contributions the EMC is the locus of possible equilibria steady-state of the economy, we have used the same name.

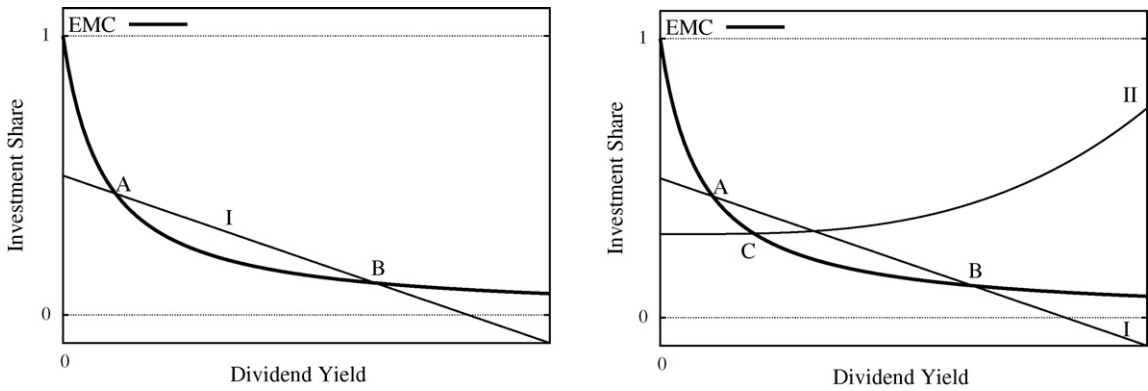


Fig. 1. Location of steady-state equilibria for $g > r_f$. **Left panel:** The Equilibrium Market Curve is shown together with one Equilibrium Investment Function, curve I. In total there are two intersections of the EIF with the EMC, A and B. Their coordinates give the equilibrium values of the dividend yield (abscissa) and of the survivor's investment share (ordinate). **Right panel:** When the EIF II is added one more steady-state arises, illustrated by point C. At the equilibria shown by A and B agent I survives, $\varphi_I = 1$, and agent II vanishes, $\varphi_{II} = 0$. Conversely, at the equilibrium illustrated by C agent II survives, $\varphi_{II} = 1$, and agent I vanishes, $\varphi_I = 0$. Since, according to Proposition 3.3, at a locally stable steady-state the investment share of the vanisher cannot lie above the investment share of the survivor, only A can be possible stable in this example.

dividend yield. Does this imply that aggressive behavior is harmful for investors' returns? The answer is no. From (3.2) one can derive the steady-state asset excess return

$$g + y^* - r_f = \frac{g - r_f}{x^*}.$$

Even if it decreases in the investment level, agent's return is given by multiple x^* of it. Hence, all the steady-states with positive yield are welfare-equivalent: at any of them, the wealth of an agent grows at a rate g .

Zero yield. When $y^* = 0$ the second equation of (3.1) implies that $k^* = r_f$, and the investment share is unambiguously determined as $x^* = f(r_f, \dots, r_f; 0, \dots, 0)$. In this case, for all g and r_f , we have a unique steady-state equilibrium,¹⁰ and since the risky and the riskless asset give the same return r_f , agent's wealth is also growing at rate r_f , irrespectively of his investment decision x^* . Our findings are summarized in the following

Proposition 3.1. *The dynamics generated by (3.1) has two types of steady-state equilibria. In the first type of steady-states, which may exist only when $g > r_f$, it holds*

$$k^* = g, \quad \text{while } x^* \text{ and } y^* \text{ solve } x^* = \tilde{f}(y^*) = l(y^*), \tag{3.5}$$

where $\tilde{f}(y)$ is the EIF and $l(y)$ is the EMC. Depending on \tilde{f} , any number of steady-states is possible. In all of them (if any) agent's wealth is growing at rate g .

In the second type of steady-state equilibria

$$k^* = r_f, \quad x^* = f(\underbrace{r_f, \dots, r_f}_L; \underbrace{0, \dots, 0}_{L+1}), \quad y^* = 0, \tag{3.6}$$

and the agent's wealth is growing at rate r_f . Such steady-state always exists and it is unique.

Proof. This is a special case of Propositions 5.2 and 5.5, when the number of agents is $N = 1$. \square

3.1.2. Local stability of steady-states

Since multiple steady-states can exist in the model, the question of their local stability becomes very important. To derive the stability conditions, the Jacobian matrix of the system has to be computed and evaluated at the steady-state.¹¹ For the local asymptotic stability it is sufficient that all $2L + 2$ eigenvalues of this Jacobian lie inside the unit circle. Upon specifying the investment behavior, one can check this condition directly. We are interested, instead, in results for general investment functions. First of all, for given parameters g and r_f the steady-states of only one of the two types can be locally stable. The following result is a special case of Proposition 5.6.

Proposition 3.2. *Consider the dynamics generated by (3.1). If $g > r_f$, then the steady-state equilibrium of the second type, described in (3.6), is unstable.*

¹⁰ Notice that the steady-state with zero dividend yield can be observed only asymptotically, since dividends are positive. The reader can find more on this point in Section 5.

¹¹ General references to the modern treatment of stability and bifurcation theory in discrete-time dynamical systems are Medio and Lines (2001) and Kuznetsov (2004).

Thus, if the dynamics of the single agent market converges to a steady-state, it converges to a zero-yield equilibrium, when $g \leq r_f$, and to a positive yield equilibrium, when $g > r_f$. Notice that this implies that in a *stable* steady-state the wealth return of an agent is equal to the *maximum* between g and r_f . Intuitively, in our economy new money (and therefore agent's wealth) is continuously arriving both through dividends and interest payments. When the economy converges to a steady state, the long-run rate of growth of the total wealth is determined by the fastest among these two sources.

Technically, the previous result holds because in the steady-state of the second type one of the eigenvalues is equal to $(1 + g)/(1 + r_f)$. Instead, at all the steady-states of the first type, which exist only when $g > r_f$, there is an eigenvalue $(1 + r_f)/(1 + g)$ which is always lower than one. As for the other $2L + 1$ eigenvalues of system (3.1), at any steady-state at most $L + 1$ of them are non-zero. They are the roots of polynomials which are derived in Section 5 in a general case; see polynomial (5.7) for the steady-states with positive yields and polynomial (5.10) for the steady-state with zero yield. The polynomials, and so their roots, depend on the partial derivatives of the investment function with respect to its arguments and also (for the steady-states of the first type) on the derivative of the slope of the EMC at the steady-state. We anticipate from the full analysis of Section 5 that, in the special case of a constant investment function, the polynomials for both steady-state types reduce to $Q(\mu) = \mu^{L+1}$, leading to zero eigenvalues and local stability. By continuity, this also implies that a steady-state is stable if the investment function is flat enough in it.

3.2. Two agents

The case of two co-existing agents is more interesting because the relative wealth dynamics starts to play a role. Using (2.7) one obtains

$$\begin{cases} x_{1,t+1} = f_1(k_t, k_{t-1}, \dots, k_{t-L+1}; y_{t+1}, y_t, \dots, y_{t-L+1}), \\ x_{2,t+1} = f_2(k_t, k_{t-1}, \dots, k_{t-L+1}; y_{t+1}, y_t, \dots, y_{t-L+1}), \\ \varphi_{1,t+1} = \varphi_{1,t} \frac{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)x_{1,t}}{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)(\varphi_{1,t}x_{1,t} + \varphi_{2,t}x_{2,t})}, \\ \varphi_{2,t+1} = \varphi_{2,t} \frac{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)x_{2,t}}{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)(\varphi_{1,t}x_{1,t} + \varphi_{2,t}x_{2,t})}, \\ k_{t+1} = r_f + \frac{\sum_{n=1}^2 ((1 + r_f)(x_{n,t+1} - x_{n,t}) + y_{t+1}x_{n,t}x_{n,t+1})\varphi_{n,t}}{x_{1,t}(1 - x_{1,t+1})\varphi_{1,t} + x_{2,t}(1 - x_{2,t+1})\varphi_{2,t}}, \\ y_{t+1} = y_t \frac{1 + g}{1 + k_t}. \end{cases} \quad (3.7)$$

A steady state of the market dynamics consists of constant values of the two investment shares (x_1^* and x_2^*), constant wealth distribution (φ_1^* and $\varphi_2^* = 1 - \varphi_1^*$), and constant levels of price returns and dividend yields (k^* and y^*). According to Definition 2.1 an agent is a survivor at this steady-state, or simply "survivor" when no confusion arises, if his steady-state wealth share is positive. Instead, an agent with zero equilibrium wealth share is a vanisher.

3.2.1. Location of steady-states

Positive yield. As before, positive y^* implies that $k^* = g$, and that steady-states may exist only when $g > r_f$. Assuming this, from the wealth dynamics we obtain that the investment share of any steady-state survivor must be equal to $\varphi_1^*x_1^* + \varphi_2^*x_2^*$. Therefore, either only one agent survives or all survivors have the same steady-state investment share. Then, two possibilities can arise. If only one agent, say the first, survives, i.e., $\varphi_1^* = 1$, his investment share and the dividend yield should simultaneously satisfy to

$$x_1^* = \tilde{f}_1(y^*) \quad \text{and} \quad x_1^* = l(y^*). \quad (3.8)$$

This is the same system of equations as (3.2). Thus, again, the EMC is the locus of possible steady-state values (y^* , x_1^*). Once investment behaviors are specified equilibrium yields and investment shares are given by the coordinates of the intersection between the EMC and the EIF of the first agent. Another possibility arises when both agents survive, i.e., $\varphi_1^* \in (0, 1)$. Then it must be

$$x_1^* = \tilde{f}_1(y^*), \quad x_2^* = \tilde{f}_2(y^*) \quad \text{with} \quad x_1^* = x_2^* = l(y^*), \quad (3.9)$$

that is, both agents have to invest the same fraction of wealth in the risky asset. Since the wealth share φ_1^* can have any value in $(0, 1)$, in this case there exists an infinite number of steady-states, a continuum manifold of them to be precise. Indeed, any wealth distribution corresponds to a different steady-state, though the levels of k^* and y^* are the same in all of them.

The right panel of Fig. 1 gives a specific example of positive yield equilibria. The EIF of agents I and II are plotted as the thin lines I and II respectively. All possible steady-states of the dynamics are the intersections between these curves and the EMC. There exist three steady-states marked by points A, B and C. Point A corresponds to the case where (3.8) is satisfied for agent I. Therefore, A illustrates a steady-state where agent I takes all the available wealth, $\varphi_1^* = 1$, i.e., I survives at A. As in the single agent case, the equilibrium dividend yield y^* is the abscissa of point A, while the investment share of the survivor at A, x_1^* , is the ordinate of A. In the other two steady-states the variables are determined in a similar way. In particular, B

characterizes a steady-state where agent I is the only survivor, while C illustrates a steady-state where the second agent is the only survivor. To illustrate a situation with infinity of steady-states, imagine to shift curve II upward so that points C and A coincide, $A \equiv C$. This would be a situation with two survivors at A. In fact when the two EIFs intersect the EMC in the same point, system (3.9) is satisfied. Any wealth distribution (i.e., arbitrary combination of φ_I and φ_{II} satisfying to $\varphi_I + \varphi_{II} = 1$) defines a steady-state. Since it is unlikely that two “generic” investment functions intersect the EMC at the same point, we refer to such case of coexisting survivors as non-generic.

Zero yield. If $y^* = 0$ we derive from (3.7) that $k^* = r_f$ and the investment shares are uniquely determined from the investment functions, i.e., $x_n^* = f_n(r_f, \dots, r_f; 0, \dots, 0)$ for $n = 1$ and 2. As opposed to the single agent case, the zero yield steady-state is not unique, because any wealth distribution determines a steady-state. In all of them, however, the aggregate economic variables (the price return and dividend yield) are the same. Therefore, such steady-states are indistinguishable when looking at the aggregate time series. They exist for any value of g and r_f .

To summarize, when two agents are in the market, there are two types of steady-states. In steady-states of the first type $k^* = g$, the yield is positive, and all the survivors have the same investment share, namely the coordinates of an intersection of the EMC with the EIF of the survivor(s). Generically there is one survivor in each steady-state, but if the EIF of the two agents intersect in a point which also belongs to the EMC, there exist an infinite number of steady-states with two survivors, each corresponding to a different wealth distribution. Proposition 5.2 generalizes this result to the N agents case. In steady-states of the second type the yield is zero, $k^* = r_f$, all investment decisions are equivalent, and any wealth distribution is allowed, so that there exists an infinite number of such steady-states. The same holds for arbitrary N as shown in Proposition 5.5.

3.2.2. Local stability

The result of Proposition 3.2 carry over to the case with two agents. Namely, $g \leq r_f$ is a necessary condition for local asymptotic stability of the zero-yield equilibria.

An important difference with respect to the single agent case concerns additional necessary condition for local stability of the positive yield equilibria. It turns out that when only one agent (say, the first) survives, the Jacobian matrix of system (3.7) has one eigenvalue equal to

$$\frac{1 + r_f + (g + y^* - r_f)x_2^*}{1 + r_f + (g + y^* - r_f)x_1^*}, \quad (3.10)$$

implying the following result.

Proposition 3.3. Consider the positive yield steady-state equilibrium of the dynamics generated by (3.7), in which $\varphi_1^* = 1$. If

$$x_1^* = \tilde{f}_1(y^*) < x_2^* = \tilde{f}_2(y^*),$$

then the steady-state is unstable.

Proof. This follows from the proof of Proposition 5.3, when $N = 2$. \square

A survivor at a locally asymptotically stable steady-state (and so an agent who survives also on all the trajectories starting in its neighborhood) cannot behave less “aggressively”, i.e., invest less in the risky asset, than the vanisher at this steady-state. This result is a consequence of the wealth-based selection. As long as the risky asset yields a higher average return than the riskless asset, the most aggressive agent has also the highest individual return. As an example, consider the steady-state where the survivor invests a smaller share than the vanisher. Deviate now from such steady-state by redistributing wealth in favor of the vanisher. As soon as this vanisher has positive wealth, he will get higher returns and a faster growing wealth. Thus, the dynamics will not return to the initial steady-state.

We can illustrate Proposition 3.3 with the help of the right panel of Fig. 1. The steady-state value of the investment share of any agent is given by the ordinate of a point on his EIF, whose abscissa is the equilibrium dividend yield. Thus, Proposition 3.3 says that at the stable equilibrium the investment share of the survivor should not lie below the investment share of the vanisher. Given the EIFs I and II, only the steady-state illustrated by A could possibly be locally stable, while the steady-states represented by B and C are necessarily unstable. For instance, in B investor I is the only survivor, but in this steady-state a vanishing agent II invests more.

As in a single agent case, the local stability property depends also on the location of the other $L + 1$ eigenvalues of the Jacobian. The coefficients of the corresponding polynomials are determined now by the weighted average of the derivatives of the investment functions, with weights given by wealth shares in a corresponding steady-state. For instance, a steady-state where two agents, one with constant investment function and the other with the function which is very responsive to the past returns and yields, co-exist, is locally stable if the first agent has most wealth and locally unstable when the second agent has most wealth.

Let us summarize the results of this section. One feature of our model, the inflow of the numéraire from dividends and riskless return, implies that the steady-states can only be stable when prices grow at a rate equal to $\max(g, r_f)$. When $g > r_f$ the dynamics are consistent with a positive dividend yield, implying a positive excess return of the risky asset. In this case the second feature of our model starts to play its role. Namely, in our CRR framework, wealth dynamics rewards more aggressive agents, those having higher investment shares. The steady-state survivor is locally stable, and thus surviving also

in a neighborhood of the steady-state, only if he is more aggressive than the steady-state vanisher. When instead $g \leq r_f$, the excess return is zero, and wealth selection is not “activated”. In fact, every investment strategy gives the same return. As a result, a variety of investment behaviors can be observed at such steady-state. In addition to these two forces, i.e., monetary expansion and wealth selection, the investment behavior itself affects the stability of the different steady-states. Some of these effects are analyzed later, but we already mentioned that the investment functions should be flat enough in a neighborhood of a stable steady-state. Otherwise, small deviations of price returns or dividend yield is amplified by agents behavior. This finding is in line with the HAM literature.

4. An example with mean-variance optimizers

As an application of the previous section, here we consider a market with one or two myopic mean-variance optimizers whose expectations are formed as an average of past observations. The purpose of this analysis is to give a concrete example of how our results can be used in practice to analyze the co-evolution of price returns, dividend yields and wealth shares in a market with few agents.

Each agent maximizes the mean-variance utility of the next period total return

$$U = E_t[x_t(k_{t+1} + y_{t+1}) + (1 - x_t)r_f] - \frac{\gamma}{2}V_t[x_t(k_{t+1} + y_{t+1})],$$

where E_t and V_t denote, respectively, the mean and the variance conditional on the information available at time t , and γ is the coefficient of risk aversion. Assuming constant expected variance $V_t = \sigma^2$, the investment fraction which maximizes U is

$$x_t = \frac{E_t[k_{t+1} + y_{t+1} - r_f]}{\gamma\sigma^2}. \quad (4.1)$$

Agents estimate the next period return as the average of L past realized returns, that is,

$$E_t[k_{t+1} + y_{t+1}] = \frac{1}{L} \sum_{\tau=1}^L (k_{t-\tau} + y_{t-\tau}). \quad (4.2)$$

Following [Assumption 2.2](#), we forbid short positions and bound the investment shares in the interval $[\underline{b}, \bar{b}] \subset (0, 1)$. An investment function is given by

$$f_{\alpha,L} = \min \left\{ \bar{b}, \max \left\{ \underline{b}, \frac{1}{\alpha} \left(\frac{1}{L} \sum_{\tau=1}^L (k_{t-\tau} + y_{t-\tau}) - r_f \right) \right\} \right\}, \quad (4.3)$$

where $\alpha = \gamma\sigma^2$ is the “normalized” risk aversion and L is the memory span. These are the two parameters that distinguish agents.

4.1. Positive yield steady-states

We have learned in the previous section that these type of steady states only exist when $g > r_f$, which we assume from now. Deriving the EIF one has

$$\tilde{f}_{\alpha}(y) = \min \left\{ \bar{b}, \max \left\{ \underline{b}, \frac{y + g - r_f}{\alpha} \right\} \right\}, \quad (4.4)$$

which is shown by the thin curve on the left panel of [Fig. 2](#). All the steady-state equilibria can be found as the intersections of the EIF with the EMC. Notice that (4.4) does not depend on L , that is the memory span does not influence the location of the steady-states. Geometrically, all the multi-dimensional investment functions differed only in L collapse onto the same EIF. Analytically the steady-state equilibria with a single investor can be derived from [Proposition 3.1](#). We obtain

Corollary 4.1. Consider the dynamics generated by (3.1) with $g > r_f$ and with an agent investing according to (4.3). There exists a unique steady-state equilibrium (x^*, k^*, y^*) . It is characterized by $k^* = g$ and $A_{\alpha} = (y^*, x^*)$ with:

$$\begin{aligned} y^* &= (1 + \bar{b})(g - r_f)/\bar{b}, & x^* &= \bar{b}, & \text{for} & & 0 < \alpha &\leq (g - r_f)/\bar{b}^2, \\ y^* &= \sqrt{\alpha(g - r_f)} - (g - r_f), & x^* &= \sqrt{\frac{g - r_f}{\alpha}}, & \text{for} & & (g - r_f)/\bar{b}^2 < \alpha < (g - r_f)/\underline{b}^2, \\ y^* &= (1 + \underline{b})(g - r_f)/\underline{b}, & x^* &= \underline{b}, & \text{for} & & \alpha \geq (g - r_f)/\underline{b}^2. \end{aligned} \quad (4.5)$$

Results in (4.5) follow from straight-forward but tedious computations. However, existence and uniqueness are clear geometrically, given the shapes of the EMC and the EIF (4.4). For extremely low (high) value of α , agent’s investment share is given by the upper (lower) bound. For intermediate α the agent invests on the increasing part of his EIF, as illustrated in the left panel of [Fig. 2](#). The point A_{α} lies at the intersection of the EIF with the EMC, and its coordinates give the equilibrium

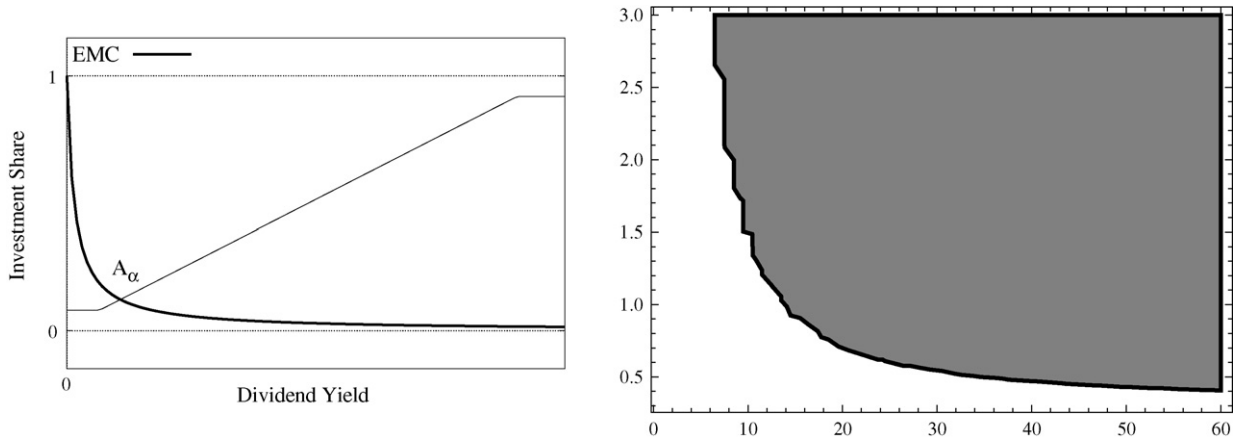


Fig. 2. Existence and stability of the steady-state equilibrium with a mean-variance optimizer, for $g > r_f$. **Left panel:** Equilibrium steady-state as the intersection of the EMC with the EIF. **Right panel:** Stability region. When a point with memory span L (as an abscissa) and normalized risk aversion α (as an ordinate) belongs to the gray area, the steady-state is stable. Growth rate of dividends and risk-free rate are, respectively, $g = 0.04$ and $r_f = 0.01$.

dividend yield, y^* , and the equilibrium investment share, x^* . The notation A_α stresses the fact that the position of this steady-state depends on the (normalized) risk aversion coefficient α but does not depend on the memory span L . It is immediate to see that when α increases, the line $x = (y + g - r_f)/\alpha$ rotates clockwise, so that the steady-state dividend yield increases, while the investment share decreases. The higher the normalized risk aversion α is, the less aggressively an agent behaves, thus lowering the equilibrium investment share and increasing the resulting equilibrium dividend yield.

What are the determinants of local stability of the steady-state equilibrium A_α ? There are $L + 1$ eigenvalues to be analyzed, which are given by the zeros of

$$Q(\mu) = \mu^{L+1} - \frac{1+g}{Ly^*} \mu^L + \frac{\mu^{L-1} + \dots + \mu}{L} + \frac{1+g+y^*}{Ly^*}, \tag{4.6}$$

a special case of the polynomial (6.2) to be derived later. The stability, therefore, is determined both by the memory span L and by the risk aversion α which enters in the polynomial via the value of y^* . By inspecting (4.6) and concentrating on the role of the memory span L the following result can be established

Corollary 4.2. For any given α the steady-state A_α is locally unstable for $L = 1$ and is locally asymptotically stable when L is big enough.

Proof. These are special cases of Propositions 6.1 and Corollary 6.1. \square

Analytical results for values of $L > 1$ are limited, but numerically one can check the stability for given parameters. In the right panel of Fig. 2 the stability region is shown as a gray area in the coordinates (L, α) . This plot confirms Corollary 4.2 and illustrates that an increase of the memory span has a stabilizing role. As we shall see in Section 6.1, this result is rather general. The intuition behind it is quite clear: the more observations an agent has, the smoother its change of the investment share in the risky asset is, thus stabilizing the dynamics.

4.2. Zero yield steady-state

At this steady-state $k^* = r_f, y^* = 0$ and the agent's investment share is $f_{\alpha,L}(r_f, \dots, r_f, 0, \dots, 0) = \underline{b}$ for any α and L . Whenever $g < r_f$, such steady-state is locally stable, since by assumption the investment function (4.3) is flat at the point $y = 0$.

4.3. Stochastic simulations

To confirm that the results above are applicable also to a stochastic system, we simulate the model with investment function $f_{\alpha,L}$ and dividends growing at a rate

$$g_t = (1 + g)\eta_t - 1,$$

where $\log(\eta_t)$ are i.i.d. normal random variables with mean 0 and variance σ_g^2 . Assumption 2.2 on the dividend process is satisfied, and in the deterministic skeleton the dividend grows with rate g . The bounds on the investment functions are taken as $\underline{b} = 0.01$ and $\bar{b} = 0.99$.

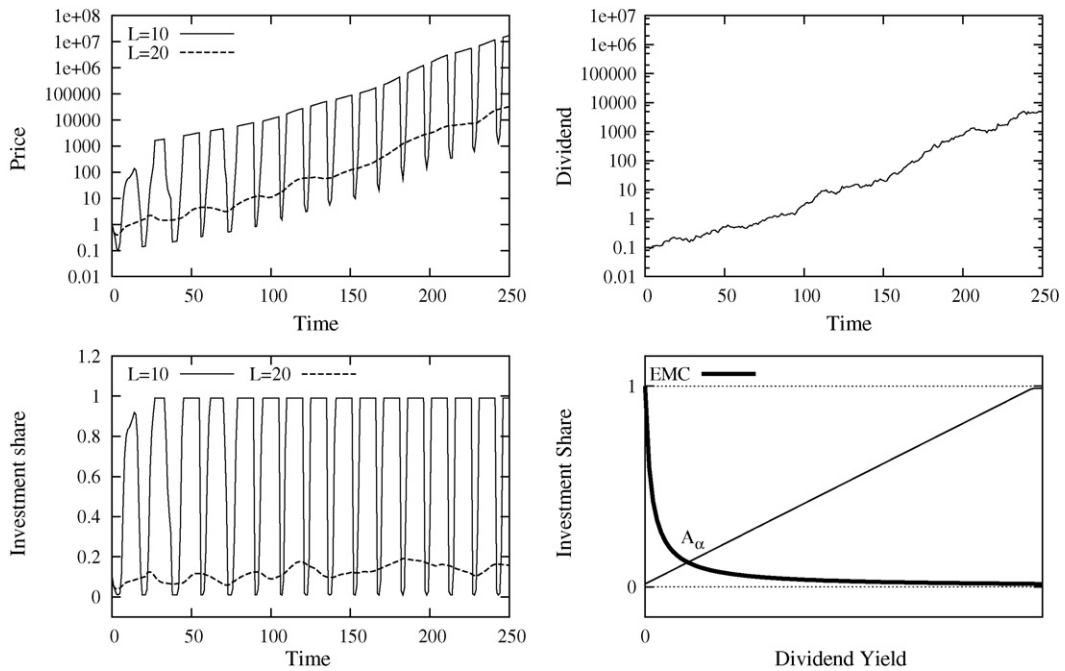


Fig. 3. Dynamics with a single mean-variance maximizer, with bounds $\underline{b} = 0.01$ and $\bar{b} = 0.99$, in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span are compared. **Top-left panel:** log-price. **Bottom-left panel:** investment share. **Top-right panel:** dividend process. **Bottom-right panel:** equilibrium on the EMC.

4.3.1. Simulations for $g > r_f$

In such a case only the steady-states with positive yield exist and may possibly be stable, see Proposition 3.1. We plot the resulting dynamics in Fig. 3. The top-right panel shows the realization of the exogenous dividend process. Given this process, simulations are performed for investment strategies with the same level of risk aversion $\alpha = 2$ and two different memory spans. The left panels show the price dynamics (top) and investment shares (bottom). When the memory span is $L = 10$ (solid line), the steady-state is unstable and prices fluctuate wildly. These endogenous fluctuations are determined by the upper and lower bounds of the investment function and are much more pronounced than the fluctuations of the exogenous dividend process. When the memory span is increased to $L = 20$ (dashed line), the system converges to the stable steady-state equilibrium and fluctuations are only due to exogenous noise affecting the dividend growth rate.

Being particularly interested in assessing the effect of wealth-driven selection, we turn now to the analysis of a market with two agents. The bottom-right panel of Fig. 4 shows the Equilibrium Investment Functions of two different agents, $\tilde{f}_{\alpha,L}$ and $\tilde{f}_{\alpha',L'}$. Specifically, we assume that the second agent has smaller risk aversion, $\alpha' < \alpha$, i.e., he is more aggressive. According to Proposition 3.3, the necessary condition for local stability of the steady-state is a non-lower investment share of the steady-state survivor. Since for any y the agent with high risk aversion, α , invests less than the agent with low risk aversion, α' , the former agent cannot survive at the stable steady-state. Thus, the steady-state represented by A_α is unstable. On the other hand, the stability of the second steady-state, $A_{\alpha'}$, depends on the local behavior of its survivor, in particular on the memory span L' . According to Corollary 4.2, if the memory span is high enough, the steady-state $A_{\alpha'}$ is locally stable. In this case the less risk averse agent is the only survivor not only at the steady-state but also on all the trajectories starting close to it.

Fig. 4 shows the market dynamics when one agent has risk aversion $\alpha = 2$ and memory $L = 20$ (which produces a stable dynamics in a single agent case, cf. Fig. 3), and the other agent has risk aversion $\alpha' = 1$ and memory L' . Simulations for two different values of the memory span L' are compared. When the memory span of the less risk averse agent is low, $L' = 20$, the steady-state $A_{\alpha'}$ is unstable (dashed lines). Wealth shares of both agents keep fluctuating between zero and one, so it seems that both agents survive on this trajectory. However, when the memory span increases to $L' = 30$, the steady-state $A_{\alpha'}$ becomes stable and the less risk averse agent is the only survivor (solid lines). The steady-state return now converges, on average, to $g + y_{\alpha'}^* < g + y_\alpha^*$. Interestingly, in our framework low risk aversion leads to survival at the cost of lowering the market return. In fact, the agent with a lower risk aversion, being the only survivor, produces lower equilibrium yield by investing a higher wealth share in the risky asset. However, consistently with Proposition 3.1, the total wealth return is g at the stable steady-state, independently from the survivor's investment strategy.

4.3.2. Simulations for $g < r_f$

We repeat the simulations we have just performed for a higher value of r_f , such that $g < r_f$. Fig. 5 confirms our previous analysis. It shows market dynamics for a single agent with memory span either $L = 10$ or $L = 20$ (cf. Fig. 3). Whereas

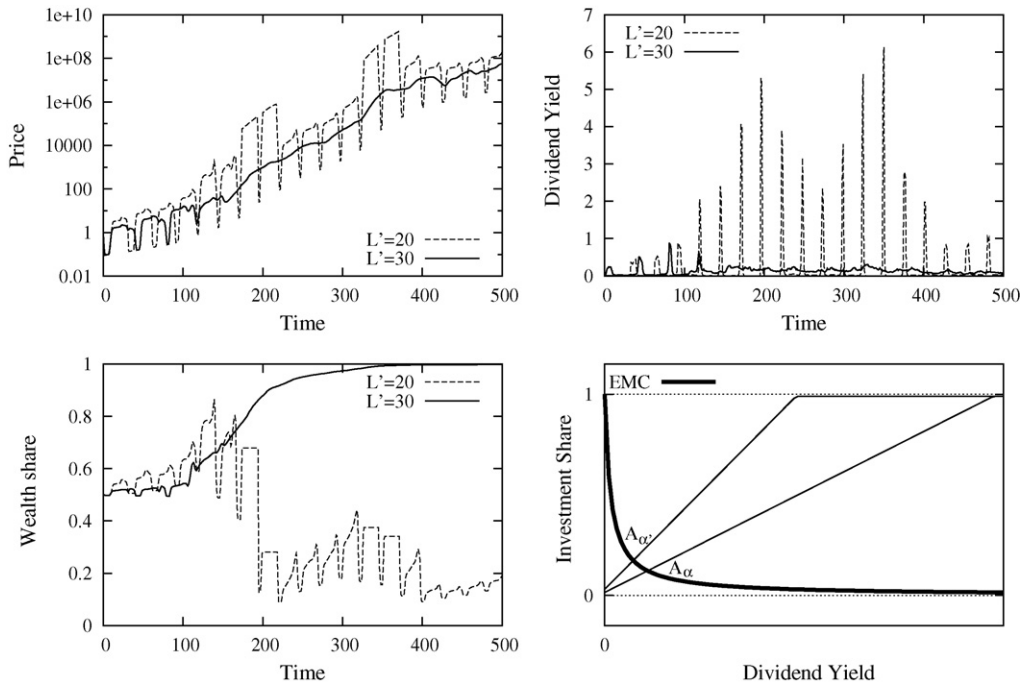


Fig. 4. Dynamics with two mean-variance maximizers, with bounds $\underline{b} = 0.01$ and $\bar{b} = 0.99$, in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span for the agent with lower risk aversion are compared. **Top-left panel:** log-price. **Bottom-left panel:** wealth share of the agent with lower risk aversion. **Top-right panel:** dividend yield. **Bottom-right panel:** EMC and two investment functions. The agent with lower risk aversion α' produces the steady state $\Delta\alpha'$.

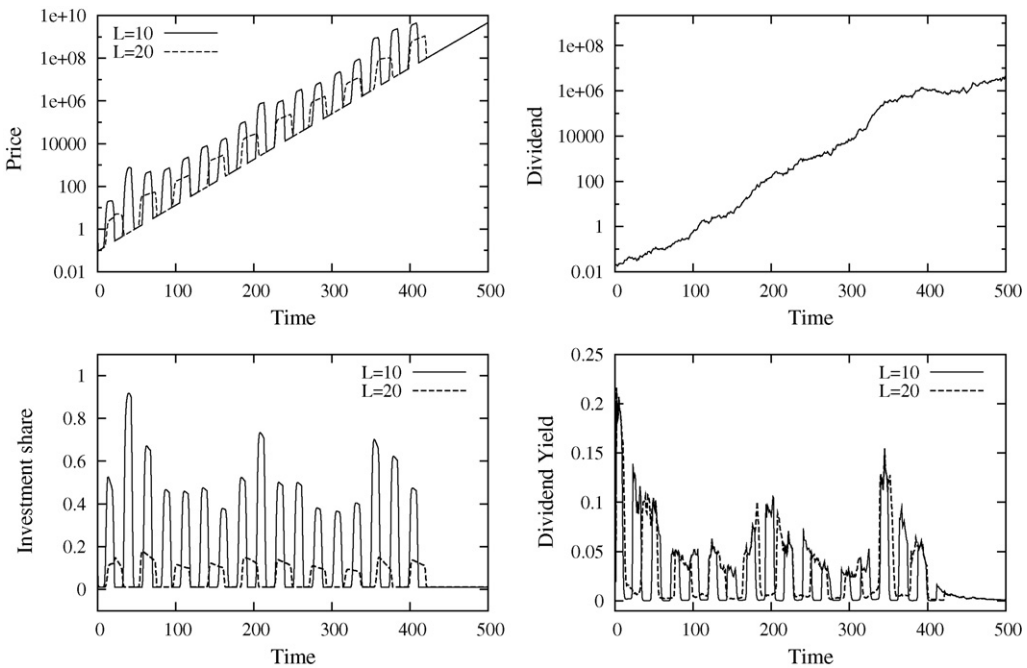


Fig. 5. Dynamics with a single mean-variance maximizer, with bounds $\underline{b} = 0.01$ and $\bar{b} = 0.99$, in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span are compared. **Top-left panel:** log-price. **Bottom-left panel:** investment share. **Top-right panel:** dividend process. **Bottom-right panel:** dividend yield.

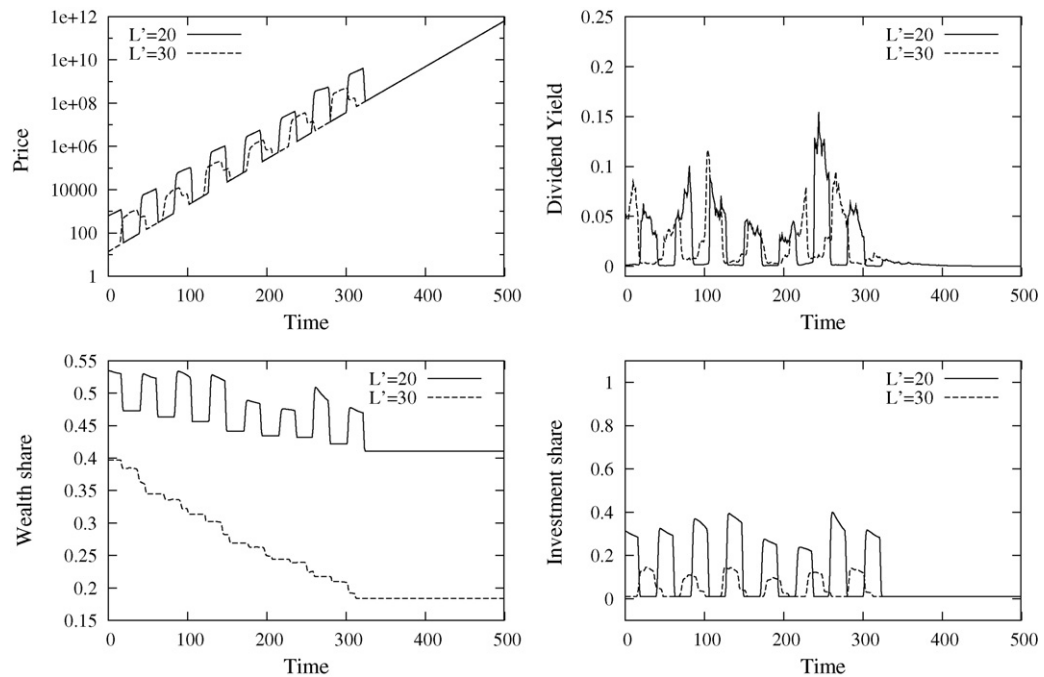


Fig. 6. Dynamics with two mean-variance maximizers, with bounds $\underline{b} = 0.01$ and $\bar{b} = 0.99$, in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span for the agent with lower risk aversion are compared. **Top-left panel:** log-price dynamics. **Bottom-left panel:** wealth share of the agent with lower risk aversion α . **Top-right panel:** dividend yield. **Bottom-right panel:** weighted average of the agents' investment shares.

with $g > r_f$ the market is stable with long memory and unstable with short memory, with $g < r_f$ the market dynamics stabilizes no matter the value of L . This is due to the presence of the lower bound $\underline{b} = 0.01$. In fact for any α and L one has $f_{\alpha,L}(r_f, \dots, r_f; 0, \dots, 0) = \underline{b}$ which is constant, so that the price feedbacks is not activated and this possible source of instability is eliminated. The price grows at the constant rate r_f (top-left panel), no matter the exogenous fluctuations of the dividend process (top-right panel). Since the price grows faster than the dividend, the dividend yield converges to 0 (bottom-right panel). At the steady-state the agent is investing a constant fraction of wealth equal to the lower bound of (4.3), i.e., $x^* = 0.01$ in this case (bottom-left panel).

Fig. 6 shows the market dynamics when two mean-variance optimizers, with different values of risk aversions, are active (cf. Fig. 4). No matter the memory span of the less risk averse agent, the price dynamics stabilizes (top-left panel). Prices grow at the constant rate r_f , despite the exogenous fluctuations of the dividend process. Since prices are growing faster than dividends, the dividend yield converges to 0 (top-right panel). At the steady-state both agents survive having positive wealth shares (bottom-left panel), and they both invest $x^* = 0.01$ (bottom-right panel). All agents are gaining the same returns and the market is not selecting among them.

5. Market dynamics with N investors

In this section we generalize and formalize results we have already encountered by addressing the equilibrium and stability analysis of the market dynamics given by (2.7) in full generality, i.e., when the market is populated by N investors each following a different investment behavior.

The primary issue is whether restricting the dynamics to the set of economically relevant values delivers a well-defined dynamical system. In particular, positivity of prices and dividends imply that price returns should always exceed -1 and dividend yields should always be larger than 0. The following result shows that at this purpose it has been sufficient to introduce Assumption 2.2, i.e., forbid short-selling.

Proposition 5.1. *The system (2.7) defines a $2N + 2L$ -dimensional dynamical system of first-order equations. Provided that Assumption 2.2 holds, the evolution operator associated with this system*

$$\mathcal{T}(x_1, \dots, x_N; \varphi_1, \dots, \varphi_N; k_1, \dots, k_L; y_1, \dots, y_L) \quad (5.1)$$

is well-defined on the set

$$\mathcal{D} = (0, 1)^N \times \Delta_N \times (-1, \infty)^L \times (0, \infty)^L, \tag{5.2}$$

consisting respectively of investment shares, wealth shares, (lagged) price returns and (lagged) dividend yields, and where Δ_N denotes the unit simplex in N -dimensional space

$$\Delta_N = \left\{ (\varphi_1, \dots, \varphi_N) : \sum_{m=1}^N \varphi_m = 1, \varphi_m \geq 0 \quad \forall m \right\}.$$

Proof. We prove that the dynamics from \mathcal{D} to \mathcal{D} is well-defined. The explicit evolution operator \mathcal{T} , which is used in the stability analysis, is provided in [Appendix A](#).

Let us start with period- t variables belonging to the domain \mathcal{D} and apply the dynamics described by (2.7) to them. Since $k_t > -1$, the fourth equation is well defined and y_{t+1} is positive. As a result, the first equation defines the new investment shares belonging to $(0, 1)$ in accordance with [Assumption 2.2](#). It, in turn, implies that in the right-hand side of the third equation all the variables are defined, and the denominator is positive. Thus, k_{t+1} can be computed. Moreover, the denominator does not exceed 1, as a convex combination of numbers non-exceeding 1. Then, a simple computation gives

$$k_{t+1} > r_f + \sum_m ((1 + r_f)(-1) + 0)\varphi_{m,t} = -1.$$

Finally, it is easy to see that both the numerator and the denominator of the second equation are positive and that $\sum_m \varphi_{m,t+1} = 1$. Therefore, the dynamics of the wealth shares is well-defined and takes place within the unit simplex Δ_N . \square

The proposition shows that given any initial conditions for (2.7), the dynamics¹² of price returns, dividend yields and relative wealth shares are completely specified. One can now easily derive the dynamics of price and wealth levels as well, but only if the initial price and the initial wealth of agents are given.

5.1. Location of steady-state equilibria

In a steady-state with M surviving agents ($1 \leq M \leq N$) we always assume that the first M agents are those who survive. The characterization of all possible steady-states of the dynamical system defined on the set \mathcal{D} is given below.

Proposition 5.2. *Steady-state equilibria of the dynamical system (2.7) evolving on the set \mathcal{D} exist only when the dividend growth rate g is larger than the interest rate r_f .*

Let $g > r_f$ and let $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$ be a steady-state of (2.7). Then:

- The steady-state price return is equal to the growth rate of dividends, $k^* = g$;
- All surviving agents have the same investment share x_{\diamond}^* , which together with the steady-state dividend yield y^* satisfy

$$x_{\diamond}^* = \frac{g - r_f}{y^* + g - r_f}. \tag{5.3}$$

- The steady-state wealth shares satisfy

$$\begin{aligned} \varphi_m^* \in (0, 1] & \quad \text{if } m \leq M \\ \varphi_m^* = 0 & \quad \text{if } m > M \end{aligned} \quad \text{and} \quad \sum_{m=1}^M \varphi_m^* = 1. \tag{5.4}$$

- The wealth return of each surviving agent is g .

Proof. See [Appendix B](#). \square

We have established that a steady-state can only exist when $g > r_f$. This result begs a question of what happens in the opposite case, when the dividend growth rate is smaller than r_f . It turns out that when $g \leq r_f$ the dynamics, if converges, does it to a point where $y^* = 0$ which, formally, does not belong to the domain \mathcal{D} defined in (5.2). We postpone the formal analysis of these situations, which we have already encountered in the examples of Section 3, to Section 5.3. For the moment just assume that $g > r_f$. Then many situations are possible, including the cases with no steady-state, with multiple steady-states, and with different number of survivors at the same steady-state.

[Proposition 5.2](#) implies that the dividend yield, y^* , and the investment share of survivors, x_{\diamond}^* , are determined simultaneously by (5.3). Using the Equilibrium Market Curve (EMC) $l(y)$ defined in (3.4) and the Equilibrium Investment Function (EIF) $\tilde{f}(y)$ defined in (3.3), equation (5.3) can be rewritten as the following system of M equations

$$l(y^*) = \tilde{f}_m(y^*) \quad \forall 1 \leq m \leq M. \tag{5.5}$$

¹² With some abuse of language we usually refer to (2.7) and not to the first order map \mathcal{T} in (5.1) as “the dynamical system”. The explicit map \mathcal{T} is used only in the proofs.

In Fig. 1 we have shown how the condition (5.5) can be expressed graphically. Namely, all possible pairs (y^*, x_\diamond^*) can be found as the intersections of the EMC with each EIF. The EMC-plot shows that the often heard conjecture that, in the world of heterogeneous agents, “anything goes” is not necessarily valid. Even when the strategies of agents are unspecified, as in our framework, the market and wealth dynamics play their role in shaping the aggregate outcome. The steady-states of system (2.7) can lie only on the EMC, which is a small subset of the original domain. The shape of the EMC is entirely determined by the exogenous parameters of the model, as g and r_f , and does not depend on agents’ behaviors.

5.2. Stability analysis

We turn now to the characterization of the local stability conditions for the steady-states found in Proposition 5.2. At this purpose we assume that all the investment functions entering into the dynamics (2.7) are differentiable at the corresponding steady-states.

5.2.1. Wealth-driven selection

The steady-states identified in Proposition 5.2 are characterized by a positive excess return, which allows the market to play the role of a natural selecting force. In fact, trading rewards some agents at the expense of others, shaping in this way the long-run wealth distribution. The first part of our stability analysis focuses on this “natural selection” mechanism. The following result establishes sufficient condition for stability and also explain the necessary condition given in Proposition 3.3.

Proposition 5.3. Consider the steady-state equilibrium $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$ described in Proposition 5.2, where the first M agents survive and invest x_\diamond^* . It is (locally asymptotically) stable if the following two conditions are met:

(1) the investment shares of the vanishing agents are such that

$$x_m^* < x_\diamond^* \quad \forall m \in \{M + 1, \dots, N\}. \tag{5.6}$$

(2) the steady-state $(x_1^* \equiv x_\diamond^*, \dots, x_M^* \equiv x_\diamond^*; \varphi_1^*, \dots, \varphi_M^*; k^*; y^*)$ of the reduced system, obtained by elimination of all the vanishers from the economy, is locally asymptotically stable.

Proof. In Appendix C we show that condition (1) is necessary and sufficient to guarantee that M eigenvalues of the Jacobian matrix lie inside the unit circle. Among the other eigenvalues there are M zeros. Finally, all the remaining eigenvalues can be derived from the Jacobian associated with the “reduced” dynamical system, i.e. without vanishing agents, evaluated in the steady-state. This implies condition (2). □

5.2.2. Stability of equilibria with steady-state survivors

According to condition (2) in Proposition 5.3, when (5.6) is satisfied, the vanishers can be eliminated from the market. The dynamics can then be described by the reduced system, that is, the same system (2.7) but with only M agents, all investing the same, as if there is a representative investor. When is the corresponding steady-state equilibrium stable? The general answer to this question is quite complicated because stability depends upon the behavior of the steady-state survivors in a small neighborhood of the steady-state, i.e., on the slopes of their investment functions.

Let $\mathbf{e}^* = (k^*, \dots, k^*; y^*, \dots, y^*)$ be the vector of lagged returns and yields at the steady-state. In the notation below index agents with $m = 1, \dots, N$ and lags with $l = 0, \dots, L - 1$. Denote the derivative of the investment function f_m with respect to the contemporaneous dividend yield as f_m^Y , the derivative with respect to the dividend yield of lag $l + 1$ as f^{y_l} , and the derivative with respect to the price return of lag $l + 1$ as f^{k_l} . Furthermore, introduce

$$(f^Y) = \sum_{m=1}^M \varphi_m^* f_m^Y(\mathbf{e}^*), \quad (f^{y_l}) = \sum_{m=1}^M \varphi_m^* f_m^{y_l}(\mathbf{e}^*), \quad (f^{k_l}) = \sum_{m=1}^M \varphi_m^* f_m^{k_l}(\mathbf{e}^*),$$

which are the weighted derivatives of the investment functions evaluated at the steady-state. Finally, $l'(y^*)$ is the slope of the EMC at the steady-state equilibrium y^* . The next result reduces the stability problem to the exploration of the roots of a certain polynomial.

Proposition 5.4. The steady-state $(x_1^*, \dots, x_M^*; \varphi_1^*, \dots, \varphi_M^*; k^*; y^*)$, described in Proposition 5.2, with M survivors is locally stable if all the roots of the polynomial

$$Q(\mu) = \mu^{L+1} - \frac{1}{l'(y^*)} \left((f^Y) \mu^L + \sum_{l=0}^{L-1} (f^{y_l}) \mu^{L-1-l} + (1 - \mu) \frac{1 + g}{y^*} \sum_{l=0}^{L-1} (f^{k_l}) \mu^{L-1-l} \right) \tag{5.7}$$

lie inside the unit circle. If, in addition, only one agent survives, then the steady-state is locally asymptotically stable. The steady-state is unstable if at least one of the roots of polynomial $Q(\mu)$ is outside the unit circle.

Proof. See Appendix D. □

When the investment functions are specified, this proposition provides a definite answer to the question about stability of a given steady-state. One has to evaluate the polynomial (5.7) in this steady-state and compute (e.g., numerically) all its $L + 1$ roots. Despite not providing explicit stability conditions, Proposition 5.4 reduces the complexity of the problem. In fact, whereas the characteristic polynomial of the dynamical system (2.7) has dimension $2N + 2L$, we are left to the analysis of a polynomial of degree $L + 1$. Proposition 5.4 shows the stability is determined by the average values of partial derivatives of the agents' investment functions weighted by the agents' equilibrium wealth shares. Hence, in the case with many survivors, some of the steady-states on the same manifold (i.e., with the same dividend yield and investment share) can be stable, while other can be unstable. As we have anticipated in Section 3, the flatter the weighted investment function is at the steady-state, i.e., the closer its partial derivatives are to zero, the more likely it is that this steady-state is stable.

5.3. Zero-yield equilibria

So far we have dealt with economies where dividends grow faster than the risk-free rate r_f . In fact, according to Proposition 5.2, only in this case there exist steady-states equilibria. What happens when dividends grow, on average, more slowly than r_f ? In this case prices tend to grow faster than dividends, so that the dividend yield goes to zero. Formally, however, $y = 0$ cannot be a point of our domain, because dividends and prices are always positive. It is now clear that the reason why in Proposition 5.2 we did not find any steady-state when $g \leq r_f$ is very simple. Since the domain \mathcal{D} given in (5.2) is not a closed set, the dynamics can easily convergence to a point which at its boundary, as $y = 0$.

Let us, therefore, extend our formal analysis of the dynamics on the set

$$\mathcal{D}' = (0, 1)^N \times \Delta_N \times [-1, \infty)^L \times [0, \infty)^L. \tag{5.8}$$

It turns out that (2.7) has a well defined dynamics also on \mathcal{D}' . In this way we are able to characterize possible asymptotic converge to a steady-state equilibrium with zero dividend yield. The next result applies.

Proposition 5.5. Consider the dynamical system (2.7) evolving on the set \mathcal{D}' introduced in (5.8) and assume that the “no-short selling” constraint of Assumption 2.2 is satisfied. Apart from the steady-state equilibria described in Proposition 5.2, the system has other steady-states equilibria $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$ where:

- The price return is equal to the risk-free rate, $k^* = r_f$.
- The dividend yield is zero, $y^* = 0$.
- The wealth shares satisfy

$$\begin{aligned} \varphi_m^* \in (0, 1] & \quad \text{if } m \leq M \\ \varphi_m^* = 0 & \quad \text{if } m > M \end{aligned} \quad \text{and} \quad \sum_{m=1}^N \varphi_m^* = 1. \tag{5.9}$$

- The wealth return of each agent is equal to r_f .

Proof. See Appendix E. \square

Contrary to the steady-state equilibria with positive dividend yield, in the steady-states derived in Proposition 5.5, the total return of the asset, $k^* + y^*$, coincides with r_f . At these steady-states, therefore, there is no difference between the return on investment of the risky and riskless asset.

The local stability of the steady-state equilibria with zero dividend yield can be analyzed along the same lines of Proposition 5.3 and leads to

Proposition 5.6. Steady states of the dynamical system (2.7) evolving on the set \mathcal{D}' can be stable only if $g \leq r_f$.

Let $g < r_f$ and let $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$ be a steady-state equilibrium of (2.7) with $y^* = 0$. This point is locally stable if all the roots of the polynomial

$$Q_0(\mu) = \mu^{L+1} + \frac{1 + r_f}{\sum_{m=1}^N x_m^* (1 - x_m^*) \varphi_m^*} (1 - \mu) \sum_{l=0}^{L-1} \left(\sum_{m=1}^N f_m^{kl}(\mathbf{e}^*) \varphi_m^* \right) \mu^{L-1-l} \tag{5.10}$$

lie inside the unit circle.

The steady-state is unstable if at least one of the roots of the polynomial $Q_0(\mu)$ is outside the unit circle.

Proof. See Appendix F. \square

6. The LLS model revisited

Having the complete picture of the market dynamics when many investors are trading we can apply our findings to shed light on the various simulations of the LLS model performed in Levy et al. (1994); Levy and Levy (1996); Levy et al. (2000); Zschischang and Lux (2001). In fact, as far as the co-evolution of prices and wealth, the demand specification, and the dividend process are concerned, the LLS model and ours coincide, as also shown in Anufriev and Dindo (2006).

In the LLS model, at period t investors maximize a power utility function $U(W_{t+1}, \gamma) = W_{t+1}^{1-\gamma}/(1-\gamma)$ with relative risk aversion $\gamma > 0$. Furthermore, to forecast the next period total return $z_{t+1} = k_{t+1} + y_{t+1}$, agents assume that any of the last L returns can occur with equal probability. In addition, a lower and an upper bound to the possible investment shares are imposed. Solutions to the maximization of a power utility are not available analytically but they have been shown to give a wealth independent investment shares. This property holds for any perceived distribution, $h(z)$, of the next period total return, which is discrete uniform in this case. Let us denote the corresponding investment function as $f^{EP}(\gamma, h(z))$, where EP stands for Expected Power. As this investment function is unavailable in explicit form, the analysis of the LLS model relies on numeric solutions. Since our results in Section 5 are valid for any functional form of the investment function, we are able to give an analytic support to the LLS model. This is possible upon characterizing the dependence of the LLS investment function on the memory span and risk aversion. We shall see that both types of dependence are similar to the case of mean-variance investors analyzed in Section 4.

6.1. Characterizing LLS investors

6.1.1. Memory span

To understand the dependence of f^{EP} on the memory span, notice that these investment functions depend on the average of past L total returns, given by the sum of price returns and dividend yields. LLS investors use this average as a forecast of future returns and invest a generic function of this forecast.¹³ Let each individual investment share be given by

$$x_{n,t} = f_n \left(\frac{1}{L} \sum_{\tau=1}^L (y_{t-\tau} + k_{t-\tau}) \right). \quad (6.1)$$

Plugging (6.1) in polynomial (5.7) and simplifying, we obtain

$$\tilde{Q}(\mu) = \mu^{L+1} - \frac{1 + \mu + \dots + \mu^{L-1}}{L} \left(1 + (1 - \mu) \frac{1+g}{y^*} \right) \frac{\sum_{m=1}^M f'_m(y^* + g) \varphi_m^*}{l'(y^*)}, \quad (6.2)$$

where f'_m denotes the derivative of the investment function with respect to the average past return. The last ratio in $\tilde{Q}(\mu)$ gives a relative slope between the survivors' "average" investment function and the EMC, both evaluated at the steady-state. The investment functions are weighted by the survivors' wealth shares in a corresponding steady-state, which we will denote as $\langle f'(y^* + g) \rangle = \sum_{m=1}^M f'_m(y^* + g) \varphi_m^*$. The stability conditions are still known only implicitly. When $L = 1$ they can be made explicit.

Corollary 6.1. Consider a steady-state equilibrium of the system (2.7) with investment functions (6.1) and lag $L = 1$, where all the non-survivors have been eliminated. The steady-state is locally stable if

$$\frac{-y^*}{1+g+y^*} < \frac{\langle f'(y^* + g) \rangle}{l'(y^*)} < \frac{y^*}{y^* + 2(1+g)}. \quad (6.3)$$

The steady-state generically exhibits flip or Neimark-Sacker bifurcation if, respectively, the right- or left-most inequality in (6.3) turns to equality.

Proof. This follows from standard conditions for the roots of second-degree polynomial to be inside the unit circle. See appendix G for the details. \square

Conditions (6.3) are illustrated in Fig. 7 in the coordinates $(y^*, \langle f' \rangle / l')$. The steady-state is stable if the corresponding point belongs to the dark-grey area, whose upper and lower borders are the values where, respectively, a flip or a Neimark-Sacker bifurcation occurs. From the diagram it is clear that the dynamics are stable for a low (in absolute value) relative slope $\langle f' \rangle / l'$ at the steady-state.

How does the stability depend on the memory span L ? A mixture of analytic and numeric tools help to reveal the behavior of the roots of polynomial (6.2) with higher L . The stability conditions for $L = 2$, derived in Appendix G, can be confronted with the $L = 1$ case, see Fig. 7. An increase of the memory span L from 1 to 2 enlarges the stability region. As agents look further back, any recent shock in price or return gets smaller impact on their behavior. For further increases of the memory span the following result holds

Proposition 6.1. Consider a steady-state of the system (2.7) with investment functions (6.1), where all the non-survivors have been eliminated. Provided that

$$\frac{\langle f'(y^* + g) \rangle}{l'(y^*)} < 1, \quad (6.4)$$

the corresponding steady-state is locally stable for high enough L .

¹³ Other investors using the same type of forecasts are the mean-variance agents of Section 4. Results here can thus be equally applied there.

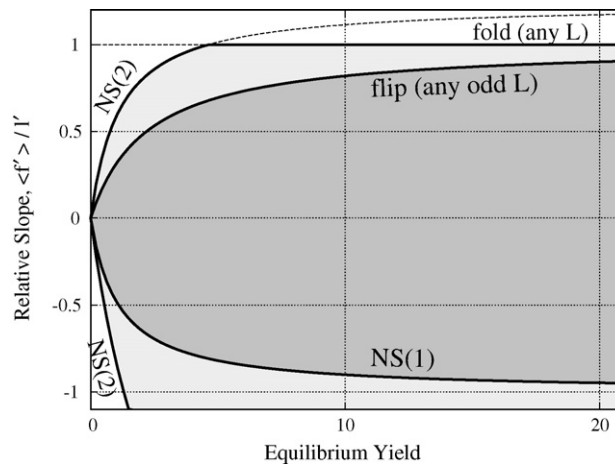


Fig. 7. Steady-state stability when investment depends on the average of past L total returns. If the pair $(y^*, (f'(y^* + g))/l'(y^*))$ belongs to the dark-grey area, the steady-state is stable for $L = 1$. When $L = 2$ the stability region expands and consists of the union of the dark-grey and light-grey areas. When $L \rightarrow \infty$ the stability region occupies all the space below “fold” line $(f'(y^* + g))/l'(y^*) = 1$. Crossing the border of the stability region causes the corresponding type of bifurcation, where NS stands for Neimark-Sacker.

What do Corollary 6.1 and Proposition 6.1 imply for the LLS model? Since l' is negative and f^{EP} is an increasing function in $z = y + k$, the ratio $(f')/l'$ is always negative. As a result, (6.4) always holds so that an increase of memory span always stabilizes the system.¹⁴

6.1.2. Risk aversion

Whereas the memory span influences the stability of the dynamics, we risk aversion determines the capability of agents to invade the market. This was already shown in the example with mean-variance maximizers of Section 4. Graphically, it holds as long as the EIF on the EMC-plot shifts upwards following a decrease in the risk aversion. This is the case also in the LLS model, since, as the following result shows (see Anufriev, 2008 for a proof), the function $f^{EP}(\gamma, h(z))$ has the same property.

Proposition 6.2. Let f_{γ}^{EP} stand for the partial derivative of the investment function f^{EP} with respect to the risk aversion coefficient γ , and \bar{z} for the expected value of the total return. Then the following result holds:

$$\text{If } \bar{z} \geq 0, \text{ then } f_{\gamma}^{EP} \geq 0 \text{ and } f_{\gamma}^{EP} \leq 0 .$$

This means that when a positive return is expected, agents with lower risk aversion invest higher shares of their wealth in the risky asset, and are thus more “aggressive”.

6.2. Interpreting LLS simulations

In Levy and Levy (1996) the focus is on the role of memory. The authors show that with a short memory span the log-price dynamics is characterized by crashes and booms. Our analysis suggests that this result is due to the presence of an unstable steady-state and of an the upper and lower bounds of the investment shares which are needed to avoid short positions and guarantee the positiveness of prices. Using Proposition 6.1 we are able to claim that this steady-state should become stable if the memory is high enough. Simulations in Levy and Levy (1996) confirm this statement. When agents with longer memory are introduced, booms and crashes disappear and price fluctuations become stationary. These are, in fact, fluctuations which are mainly due to the stationary exogenous noise of the dividend process, and not to agents’ interactions.

Some other simulations in Levy and Levy (1996) are performed with positive risk-free rate and zero dividend growth rate (i.e., $g = 0$). These simulations do converge, irrespectively of the noisy dividend. To understand why and where they converge, recall our general analysis for the case $g < r_f$, see Propositions 5.5 and 5.6. We have shown that prices are always growing at a rate r_f , no matter the initial set of investment strategies, and the dividend yield converges to $y^* = 0$. As a result wealth return is r_f for any investment strategy, and no selection on the set of investment strategies occurs. This is exactly what happens in the simulations.

In Zschischang and Lux (2001) the focus is on the interplay between the length of the memory span and the risk aversion. Their simulations suggest that the risk aversion is more important than the memory span in the determination of the

¹⁴ Notice that this is not always the case in the HAM literature. See, e.g., Chiarella et al. (2006b,c) built in a CARA framework with fundamentalists and technical traders.

dominating agents, providing that the memory is not too short. The argument has not been put forward in a decisive way though, as the following quote from Zschischang and Lux (2001, p. 568, 569) shows:

“Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion. [...] It also appears that when adding different degrees of risk aversion, the differences of time horizons are not decisive any more, provided the time horizon is not too short.”

Our analytic results make clear how and why this is the case. As shown by [Proposition 6.2](#), risk aversion is related to the agents' “aggressiveness”. This together with [Proposition 5.3](#) shows that agents with low risk aversion are indeed able to destabilize the market populated by agents with high risk aversion. However, this “invasion” leads to an ultimate domination only if the invading agents have sufficiently long memory. Otherwise, and this complements the conclusions of [Zschischang and Lux \(2001\)](#) and related works, the dynamics does not converge to any steady-state, and agents with different risk aversion coefficients coexist.

Another new result concerns the case of agents investing a constant fraction of wealth. In [Zschischang and Lux \(2001\)](#) the authors claim that such agents always dominate the market and add (p. 571):

“Hence, the survival of such strategies in real-life markets remains a puzzle within the Levy, Levy and Solomon microscopic simulation framework as it does within the Efficient Market Theory.”

Our analysis allows to make this statement more precise. The agents with constant investment fraction are characterized by the horizontal investment functions, for which [Proposition 5.4](#) guarantees stability, independently of L . If these agents are able to invade the market successfully, they will ultimately dominate. However, their market invasion will fail, as soon as other agents are more aggressive in the steady-states created by invaders.

7. Conclusion

In his recent survey, [LeBaron \(2006\)](#) stresses that agent-based models do not require analytical tractability (as opposed to Heterogeneous Agents Models) and, therefore, are more flexible and realistic for what concerns their assumptions. In this paper we show that flexibility can be achieved in an analytically-tractable heterogeneous agent framework too. In fact, we have performed an analytic investigation of a stylized model of a financial market where an arbitrary set of investors is trading. Under the assumption that the impact of different agents on the market depends on their wealth shares, we have derived existence and stability results for a general set of investment functions and an arbitrary number of agents. Due to the selecting role of wealth dynamics, we have been able to characterize the steady-state equilibria of this economy. They can either lead to different average return for the risky and the riskless assets, in which case they are at the intersection of each agent Equilibrium Investment Function with the so called Equilibrium Market Curve, or lead to the same average return for the two assets. We have also shown that our analysis of a deterministic market dynamics is helpful for the understanding of its simulations with a stochastic dividend process.

Having assumed exogenous growth dividend process, we have reached two research objectives. First, we have been able to investigate which features of previous results using the same framework, such as [Anufriev et al. \(2006\)](#) or [Anufriev and Bottazzi \(2009\)](#), are due to their assumption of a constant dividend yield and which are of a more general nature. Second, we have provided an analytical support of the LLS simulations, which would have not be possible within the CRRA framework developed so far. As for the first objective, we have shown that the specification of the dividend process does play a role in shaping the price and dividend dynamics. In our framework, the prices grow at a rate that is derived from the exogenous parameters, while the ecology of trading behaviors can affect only the dividend yield. In previous CRRA contributions on the other hand, the dividend yield is fixed and the ecology of behaviors affects the growth rate of prices and, consequently, dividends. At the same time the common CRRA framework, with a wealth-driven selection and a coupled price-wealth dynamics, is responsible for similarities such as the existence of a low-dimensional locus of possible steady-states, i.e., Equilibrium Market Curve (though of a different shape).

On the way to our second objective, we have considered an example with mean variance maximizers characterized by two parameters: degree of risk aversion and memory span used to estimate future returns. When the growth rate of dividends is larger than the risk-free rate, the agents with the lowest risk aversion dominate the market, provided that their memory spans are big enough. As a result the market dynamics converge to the stable steady-state equilibrium, where prices are growing as fast as the dividends, and lower the risk aversion the higher the value of the dividend yield. In this case price fluctuations are due to the exogenous fluctuations of dividends. Otherwise, when the memory is not long enough, agents with different investment strategies coexist and the price fluctuations are endogenously determined. When, instead, the growth rate of dividends is smaller than the risk-free rate, steady-state equilibrium asset returns are equal to risk-free returns and the dividend yield converges to zero, no matter the ecology of agents. As a result wealths returns are equal for all investment functions and there is no selection. We have also explained that these differences in the market dynamics and selecting regime are due to the exogenous inflow of wealth in the economy, both through dividends and riskless returns.

Appendix A. Dynamical system defined in Proposition 5.1

After Proposition 5.1 we have shown that the system of equations in (2.7) leads to a well-defined map from the domain \mathcal{D} , specified in (5.2), to itself. Here we explicitly provide the evolution operator of the first-order dynamical system of $2N + 2L$ variables.

First, we introduce some notation. Let us define the investment decision weighted with relative wealth as

$$\langle x_t \rangle_s = \sum_{n=1}^N x_{n,t} \varphi_{n,s}, \tag{A.1}$$

where the time of the decision, t , and the time of the weighting wealth distribution, s , can be different. Let us use the following notation for time t variables

$$x_{n,t}, \varphi_{n,t} \quad \forall n \in \{1, \dots, N\} \quad \text{and} \quad k_{t,l}, y_{t,l} \quad \forall l \in \{0, \dots, L-1\}, \tag{A.2}$$

where $k_{t,l}$ and $y_{t,l}$ denote the price return and the dividend yield at time $t - l$, respectively. We order the equations in four separated blocks: \mathcal{X} , \mathcal{W} , \mathcal{K} and \mathcal{Y} . They define, respectively, N investment choices, N wealth shares, L price returns and L dividend yields. The last two blocks are needed to update the lagged variables. The map \mathcal{T} referred in (5.1) is given by

$$\begin{aligned} \mathcal{X}: & \begin{cases} x_{1,t+1} = f_1(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}) \\ \vdots \\ x_{N,t+1} = f_N(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}) \end{cases} \\ \mathcal{W}: & \begin{cases} \varphi_{1,t+1} = \Phi_1(x_{1,t}, \dots, x_{N,t}; \varphi_{1,t}, \dots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0}); \\ \quad K[f_1(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}), \dots, \\ \quad \quad f_N(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}); \\ \quad \quad x_{1,t}, \dots, x_{N,t}; \varphi_{1,t}, \dots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})]) \\ \vdots \\ \varphi_{N,t+1} = \Phi_N(x_{1,t}, \dots, x_{N,t}; \varphi_{1,t}, \dots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0}); \\ \quad K[f_1(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}), \dots, \\ \quad \quad f_N(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}); \\ \quad \quad x_{1,t}, \dots, x_{N,t}; \varphi_{1,t}, \dots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})]) \end{cases} \\ \mathcal{K}: & \begin{cases} k_{t+1,0} = K[f_1(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}), \dots, \\ \quad \quad f_N(k_{t,0}, \dots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \dots, y_{t,L-1}); \\ \quad \quad x_{1,t}, \dots, x_{N,t}; \varphi_{1,t}, \dots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})] \\ k_{t+1,1} = k_{t,0} \\ \vdots \\ k_{t+1,L-1} = k_{t,L-2} \end{cases} \\ \mathcal{Y}: & \begin{cases} y_{t+1,0} = Y(y_{t,0}, k_{t,0}) \\ y_{t+1,1} = y_{t,0} \\ \vdots \\ y_{t+1,L-1} = y_{t,L-2}, \end{cases} \end{aligned} \tag{A.3}$$

where the function

$$Y(y, k) = y \frac{1+g}{1+k} \tag{A.4}$$

gives the dividend yield as a function of past realization of the yield and return, as in the right-hand side of the fourth equation in (2.7). The function

$$K[z_1, \dots, z_N; x_1, \dots, x_N; \varphi_1, \dots, \varphi_N; \mathcal{Y}] = r_f + \frac{(1+r_f) \sum_{m=1}^N (z_m - x_m) \varphi_m + y \sum_{m=1}^N x_m z_m \varphi_m}{\sum_{m=1}^N x_m (1 - z_m) \varphi_m} \tag{A.5}$$

gives the price return as a function of the investment choices, wealth shares and the dividend yield as in the right-hand side of the third equation in (2.7). Finally,

$$\Phi_n(x_1, \dots, x_N; \varphi_1, \dots, \varphi_N; y; k) = \varphi_n \frac{1 + r_f + (k + y - r_f)x_n}{1 + r_f + (k + y - r_f)\sum_{m=1}^N x_m \varphi_m} \quad \forall n \in \{1, \dots, N\} \tag{A.6}$$

specifies the wealth share of agent n as a function of the investment choices, wealth shares, the dividend yield and price return¹⁵ as in the right-hand side of the second equation in (2.7).

Appendix B. Proof of Proposition 5.2

To solve for the equilibrium of the system (2.7), one can substitute the time variables with equilibrium values and solve the resulting system for $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$. The system to be solved is as follows

$$\begin{cases} x_n^* = f_n(k^*, \dots, k^*; y^*, \dots, y^*), & \forall n \in \{1, \dots, N\}, \\ \varphi_n^* = \varphi_n^* \frac{(1 + r_f) + (k^* + y^* - r_f)x_n^*}{(1 + r_f) + (k^* + y^* - r_f)(x^*)}, & \forall n \in \{1, \dots, N\}, \\ k^* = r_f + \frac{y^*(x^{*2})}{(x^*(1 - x^*))}, \\ y^* = y^* \frac{1 + g}{1 + k^*}. \end{cases} \tag{B.1}$$

Since y^* and investment shares are positive, from the third equation $k^* > r_f$, while the fourth equation fixes k^* to g . Thus, equilibria exist only when $g > r_f$. In particular, it means that $k^* + y^* - r_f > 0$. The equations for the wealth shares imply that every surviving agent invests $x_n^* = (x^*)$, which is independent of n . Therefore, all the survivors invest the same share, x_\diamond^* . Plugging this share into the third equation, one gets (5.3).

Appendix C. Proof of Proposition 5.3

We denote a steady-state of the system (A.3) as

$$\mathbf{x}^* = (x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; \underbrace{k^*, \dots, k^*}_L; \underbrace{y^*, \dots, y^*}_L).$$

To derive the stability conditions for different equilibria, the Jacobian matrix has to be computed. The Jacobian depends on the derivatives of the functions Y, K and Φ_n introduced in Appendix A. We compute now the derivatives of these functions with respect to different arguments and evaluate them at each steady-state \mathbf{x}^* . For the function Y introduced in (A.4) the derivatives are given by

$$Y^y = \frac{\partial Y}{\partial y} = \frac{1 + g}{1 + k^*}, \quad Y^k = \frac{\partial Y}{\partial k} = -y^* \frac{1 + g}{(1 + k^*)^2}. \tag{C.1}$$

For the function K introduced in (A.5), for all $1 \leq m \leq N$, we have

$$\begin{aligned} K^{z_m} &= \frac{\partial K}{\partial z_m} = \varphi_m^* \frac{1 + r_f + (k^* + y^* - r_f)x_m^*}{(x^*(1 - x^*))}, \\ K^{x_m} &= \frac{\partial K}{\partial x_m} = \varphi_m^* \frac{-1 - k^* + (k^* + y^* - r_f)x_m^*}{(x^*(1 - x^*))}, \\ K^{\varphi_m} &= \frac{\partial K}{\partial \varphi_m} = x_m^* \frac{r_f - k^* + (k^* + y^* - r_f)x_m^*}{(x^*(1 - x^*))}, \\ K^y &= \frac{\partial K}{\partial y} = \frac{(x^{*2})}{(x^*(1 - x^*))}. \end{aligned} \tag{C.2}$$

¹⁵ Notice that since the sum of the wealth shares is equal to 1 at any period, one of the equations in the system (e.g. the last equation of the block \mathcal{W}) is redundant and the dynamics can be fully described by the system of dimension $2N + 2L - 1$. However, the computations are more symmetric when the relation $\varphi_{N,t} = 1 - \sum_{m=1}^{N-1} \varphi_{m,t}$ is not taken into account explicitly.

Finally, for the function Φ_n introduced in (A.6) and for all $1 \leq m \leq N$, we have

$$\begin{aligned} \Phi_n^{x_m} &= \frac{\partial \Phi_n}{\partial x_m} = (k^* + y^* - r_f) \frac{(\delta_n^m - \varphi_m^*) \varphi_n^*}{1 + r_f + (k^* + y^* - r_f)(x^*)}, \\ \Phi_n^{\varphi_m} &= \frac{\partial \Phi_n}{\partial \varphi_m} = \frac{\delta_n^m(1 + r_f) + (k^* + y^* - r_f)(\delta_n^m x_n^* - \varphi_n^* x_m^*)}{1 + r_f + (k^* + y^* - r_f)(x^*)}, \\ \Phi_n^y &= \frac{\partial \Phi_n}{\partial y} = \varphi_n^* \frac{x_n^* - (x^*)}{1 + r_f + (k^* + y^* - r_f)(x^*)}, \\ \Phi_n^k &= \frac{\partial \Phi_n}{\partial k} = \varphi_n^* \frac{x_n^* - (x^*)}{1 + r_f + (k^* + y^* - r_f)(x^*)}, \end{aligned} \tag{C.3}$$

where δ_n^m is the Kronecker's delta. Using the block structure introduced in Appendix A, the Jacobian can be written in general form as:

$$J = \begin{pmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} & \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} & \frac{\partial \mathcal{W}}{\partial \mathcal{K}} & \frac{\partial \mathcal{W}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{K}}{\partial \mathcal{X}} & \frac{\partial \mathcal{K}}{\partial \mathcal{W}} & \frac{\partial \mathcal{K}}{\partial \mathcal{K}} & \frac{\partial \mathcal{K}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{W}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{K}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} \end{pmatrix}. \tag{C.4}$$

The block $\partial \mathcal{X} / \partial \mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of agents' present investment choices with respect to agents' past investment choices. Since the investment choice of any agent does not explicitly depend on the investment choices in the previous period

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{X}} \right]_{n,m} = 0, \quad 1 \leq n, m \leq N,$$

and this block is a zero matrix. The block $\partial \mathcal{X} / \partial \mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. This is also a zero matrix since investment choices do not depend on wealth fractions by assumption:

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{W}} \right]_{n,m} = 0, \quad 1 \leq n \leq N, \quad 1 \leq m \leq N.$$

The block $\partial \mathcal{X} / \partial \mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of agents' investment choices with respect to past price returns. Let us introduce a special notation for partial derivatives of the investment functions:

$$\frac{\partial f_n}{\partial k_{t-l}} = f_n^{k_l}, \quad \frac{\partial f_n}{\partial y_{t+1}} = f_n^y, \quad \frac{\partial f_n}{\partial y_{t-l}} = f_n^{y_l}, \quad 1 \leq n \leq N, \quad 0 \leq l \leq L-1.$$

Then

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{K}} \right]_{n,l} = \begin{cases} f_n^{k_0} + f_n^y \cdot Y^k & \text{for } l = 0 \text{ (the first column)} \\ f_n^{k_l} & \text{otherwise.} \end{cases}$$

The block $\partial \mathcal{X} / \partial \mathcal{Y}$ is a $N \times L$ matrix containing partial derivatives of the agents' investment choices with respect to the past dividend yield. This block is given by

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{Y}} \right]_{n,l} = \begin{cases} f_n^{y_0} + f_n^y \cdot Y^y & \text{for } l = 0 \text{ (the first column)} \\ f_n^{y_l} & \text{otherwise.} \end{cases}$$

The block $\partial \mathcal{W} / \partial \mathcal{X}$ is $N \times N$ matrix containing the partial derivatives of agents' wealth shares with respect to agents' investment choices. It holds

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}} \right]_{n,m} = \Phi_n^{x_m} + \Phi_n^k \cdot K^{x_m}, \quad 1 \leq n, m \leq N.$$

The block $\partial \mathcal{W} / \partial \mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of agents' wealth shares with respect to agents' wealth shares. It holds

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}} \right]_{n,m} = \Phi_n^{\varphi_m} + \Phi_n^k \cdot K^{\varphi_m}, \quad 1 \leq n, m \leq N.$$

The block $\partial\mathcal{W}/\partial\mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of agents' wealth shares with respect to lagged returns. For $1 \leq n \leq N$ and $0 \leq l \leq L - 1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{K}} \right]_{n,l} = \begin{cases} \Phi_n^k \cdot (\sum_m K^{Zm} (f_m^{k0} + f_m^Y Y^k) + K^Y Y^k) + \Phi_n^Y \cdot Y^k, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{Zm} f_m^{kl} & \text{otherwise.} \end{cases}$$

The block $\partial\mathcal{W}/\partial\mathcal{Y}$ is a $N \times L$ matrix containing the partial derivatives of agents' wealth shares with respect to lagged dividend yields. For $1 \leq n \leq N$ and $0 \leq l \leq L - 1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{Y}} \right]_{n,l} = \begin{cases} \Phi_n^k \cdot (\sum_m K^{Zm} (f_m^{y0} + f_m^Y Y^y) + K^Y Y^y) + \Phi_n^Y \cdot Y^y, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{Zm} f_m^{yl} & \text{otherwise.} \end{cases}$$

The block $\partial\mathcal{K}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of lagged returns with respect to agents' investment choices. Its structure is simple, since only the first line can contain non-zero elements. It reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{X}} \right]_{l,n} = \begin{cases} K^{Xn} & \text{for } l = 0 \text{ (the first row)} \\ 0 & \text{otherwise} \end{cases}, \quad 0 \leq l \leq L - 1, \quad 1 \leq n \leq N.$$

The block $\partial\mathcal{K}/\partial\mathcal{W}$ is the $L \times N$ matrix containing the partial derivatives of lagged returns with respect to agents' wealth shares. It also has $L - 1$ zero rows and reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{W}} \right]_{l,n} = \begin{cases} K^{\varphi n} & \text{for } l = 0 \text{ (the first row)} \\ 0 & \text{otherwise} \end{cases}, \quad 0 \leq l \leq L - 1, \quad 1 \leq n \leq N.$$

The block $\partial\mathcal{K}/\partial\mathcal{K}$ is the $L \times L$ matrix containing the partial derivatives of lagged returns with respect to themselves. It has a typical structure for such matrices involving lagged prices and has 1 s under the main diagonal

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{K}} \right] = \begin{pmatrix} \sum K^{Zm} (f_m^{k0} + f_m^Y Y^k) + K^Y Y^k & \sum K^{Zm} f_m^{k1} & \dots & \sum K^{Zm} f_m^{kL-2} & \sum K^{Zm} f_m^{kL-1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The block $\partial\mathcal{K}/\partial\mathcal{Y}$ is the $L \times L$ matrix containing the partial derivatives of lagged returns with respect to lagged dividend yields. It is given by

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{Y}} \right] = \begin{pmatrix} \sum K^{Zm} (f_m^{y0} + f_m^Y Y^y) + K^Y Y^y & \sum K^{Zm} f_m^{y1} & \dots & \sum K^{Zm} f_m^{yL-2} & \sum K^{Zm} f_m^{yL-1} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The block $\partial\mathcal{Y}/\partial\mathcal{X}$ is a $L \times N$ matrix containing the partial derivatives of lagged dividend yields with respect to agents' investment choices. This block is a zero matrix

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{X}} \right]_{l,m} = 0, \quad 0 \leq l \leq L - 1, \quad 1 \leq m \leq N.$$

The block $\partial\mathcal{Y}/\partial\mathcal{W}$ is a $L \times N$ matrix containing the partial derivatives of lagged dividend yields with respect to agents' wealth shares. This is also a zero matrix and

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{W}} \right]_{l,m} = 0, \quad 0 \leq l \leq L - 1, \quad 1 \leq m \leq N.$$

The block $\partial\mathcal{Y}/\partial\mathcal{K}$ is a $L \times L$ matrix containing the partial derivatives of lagged dividend yields with respect to past price returns. The only non-zero element of this matrix is in the upper left corner, i.e.

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{K}} \right]_{l,j} = \begin{cases} Y^k & \text{for } l = j = 0 \text{ (the first row, the first column)} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the block $\partial y/\partial y$ is a $L \times L$ matrix containing the partial derivatives of lagged dividend yields with respect to themselves. This matrix is given by

$$\left[\frac{\partial y}{\partial y} \right] = \begin{bmatrix} Y^y & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

With all these definitions, one obtains the following statement about the Jacobian in the equilibria of the system.

Lemma Appendix C.1. *Let x^* be an equilibrium of system (2.7) described in Prop. 5.2 and let first M agents survive at this equilibrium. The corresponding Jacobian matrix, $J(x^*)$, has the following structure, where the actual values of non-zero elements, denoted by the symbols $*$, are varying.*

$$\begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & [& \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & & [& \frac{\partial \mathcal{X}}{\partial y} & & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\ * & \dots & * & 0 & \dots & 0 & * & \dots & * & * & \dots & * & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ [& \Phi^x &] & \vdots & \ddots & \vdots & [& \Phi_S^\varphi &] & [& \Phi_{NS}^\varphi &] & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ * & \dots & * & 0 & \dots & 0 & * & \dots & * & * & \dots & * & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & * & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ * & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & \dots & * & * & * & * & \dots & * & * \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{-y^*}{1+g} & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

The solid lines divide the matrix in the same 16 blocks as in (C.4). In addition, the first and the second column-blocks as well as the second row-block, are split into the two parts of sizes M and $N - M$. They correspond to the survivors and the non-survivors, respectively.

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial \mathcal{X}/\partial \mathcal{X}$ and $\partial \mathcal{X}/\partial \mathcal{W}$, are always zero. Two other blocks, $\partial \mathcal{X}/\partial \mathcal{K}$ and $\partial \mathcal{X}/\partial y$, in general contain non-zero elements and simplified, because in the equilibrium $k^* = g$ and therefore $Y^y = 1$ and $Y^k = -y^*/(1 + g)$.

To simplify the second row-block, notice that $\Phi_n^k = \Phi_n^y = 0$ at this equilibrium. Indeed, the numerators of the corresponding general expressions in (C.3) are 0 because all the survivors invest the same share in the equilibrium (i.e. $x_n^* = \langle x^* \rangle = x_\diamond^*$ for all $n \leq M$), while for the non-survivors $\varphi_n^* = 0$. This immediately implies that the two last blocks, $\partial \mathcal{W}/\partial \mathcal{K}$ and $\partial \mathcal{W}/\partial y$, contain only zero elements. Furthermore, from the Equilibrium Market Curve relation (5.3) we get

$$1 + r_f + (g + y^* - r_f)x_\diamond^* = 1 + g. \tag{C.5}$$

Thus, at the equilibrium steady-state

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}} \right]_{n,m} = \Phi_n^{x_m} = \begin{cases} \varphi_n^*(\delta_n^m - \varphi_m^*)(y^* + g - r_f)/(1 + g) & \text{for } n \leq M \text{ (agent } n \text{ survives)} \\ 0 & \text{otherwise,} \end{cases}$$

and all the rows corresponding to the non-survivors in this block are zero. Moreover, all the columns corresponding to the non-survivors contain only zero elements as well, since then $\delta_n^m = \varphi_m^* = 0$. We denote as Φ^x the remaining (non-zero) part of the block $\partial\mathcal{W}/\partial\mathcal{X}$. The simplifications in the next block lead to

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}} \right]_{n,m} = \Phi_n^{\varphi_m} = \begin{cases} \delta_n^m - \varphi_n^*(g - r_f)/(1 + g) & \text{for } n, m \leq M \\ -\varphi_n^*x_m^*(y^* + g - r_f)/(1 + g) & \text{for } n \leq M, m > M \\ \delta_n^m(1 + r_f + x_n^*(y^* + g - r_f))/(1 + g) & \text{for } n > M. \end{cases} \quad (\text{C.6})$$

The block of the elements from the first line of the previous expression is denoted as Φ_S^φ ; the block of the elements of the second line is denoted as Φ_{NS}^φ ; while the block of the elements from the third line (i.e. when $n > M$) only the diagonal elements are non-zero.

It is obvious that in the next row-block with L rows the elements are zero in all the lines but the first. The only exception from this rule are the elements below the main diagonal in the block $\partial\mathcal{K}/\partial\mathcal{K}$ which are all equal to 1. To compute the elements in the first row consider the derivatives of function K derived in (C.2). For the first block, $\partial\mathcal{K}/\partial\mathcal{X}$, notice that for the non-surviving agents $K^{xm} = 0$, while for the survivors, i.e. for $m \leq M$

$$K^{xm} = -\varphi_m^* \frac{1 + r_f}{(1 - x_\diamond^*)x_\diamond^*}. \quad (\text{C.7})$$

Analogously, in the next block, $\partial\mathcal{K}/\partial\mathcal{W}$, for all the survivors $K^{\varphi m} = 0$, while for all other agents $m > M$ the elements are given by

$$K^{\varphi m} = x_m^* \frac{r_f - g + x_m^*(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*} = x_m^* \frac{(x_m^* - x_\diamond^*)(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*}, \quad (\text{C.8})$$

where (C.5) was used to derive the last equality.

The simplifications in the blocks $\partial\mathcal{K}/\partial\mathcal{K}$ and $\partial\mathcal{K}/\partial\mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{zm} for all the non-survivors are zeros, while for the survivors ($m \leq M$) they are given by

$$K^{zm} = \varphi_m^* \frac{1 + g}{(1 - x_\diamond^*)x_\diamond^*}. \quad (\text{C.9})$$

Thus, all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. \square

The rest of the proof of the Proposition is now clear. Consider the Jacobian matrix derived in Lemma C.1. The last $N - M$ columns of the left column-block contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N - M$. This eigenvalue does not affect stability. Moreover, these columns and the corresponding rows can be eliminated from the Jacobian. Analogously, in each of the last $N - M$ rows in the second row-block the only non-zero entries belong to the main diagonal. Consequently, $\Phi_n^{\varphi n}$ for $n > M$ are the eigenvalues of the matrix, with multiplicity (at least) one, and the rows (together with the corresponding columns) can be eliminated from the Jacobian. Using the third line of (C.6) we get the following $N - M$ eigenvalues

$$\mu_n = \frac{1 + r_f + x_n^*(y^* + g - r_f)}{1 + g} = \frac{1 + r_f + (g - r_f)(x_n^*/x_\diamond^*)}{1 + g}$$

where the last equality follows from (C.5). Recall that the equilibria we consider, exist only when $g > r_f$. Then, with a bit of algebra, the stability conditions $-1 < \mu_n < 1$ can be simplified to conditions (5.6).

Finally, notice that the elimination of the rows and columns which we have performed reduce the Jacobian to the shape which correspond to the Jacobian of the same system in the same equilibrium but without non-surviving agents.

Appendix D. Proof of Proposition 5.4

Let us proceed with a reduced Jacobian obtained from the matrix in Lemma C.1 after eliminating the rows and columns corresponding to the survivors. We denote this Jacobian as \mathbf{L} , and an identity matrix of the same dimension $(2M + 2L) \times (2M + 2L)$ as \mathbf{I} . Then the characteristic polynomial whose roots are the eigenvalues of \mathbf{L} is the determinant $\det(\mathbf{L} - \mu\mathbf{I})$. First, we analyze it and then we identify new eigenvalues.

Let us look at the second column block of the size M in this determinant. The only non-zero elements in this block lie in the rows of the second row block, in the part which was called $[\Phi_S^\varphi]$. The elements of this part have been computed in the first line in (C.6). Thus, this column block can be represented as $| \nu \mathbf{b} + \mathbf{b}_1 | \dots | \nu \mathbf{b} + \mathbf{b}_k |$, where $\nu = (g - r_f)/(1 + g)$ and

the following column vectors have been introduced

$$\begin{aligned}
 \mathbf{b} &= \parallel 0 \dots 0 \mid -\varphi_1^* \dots -\varphi_M^* \mid 0 \ 0 \dots 0 \mid 0 \ 0 \dots 0 \parallel, \\
 \mathbf{b}_1 &= \parallel 0 \dots 0 \mid 1-\mu \dots 0 \mid 0 \ 0 \dots 0 \mid 0 \ 0 \dots 0 \parallel, \\
 &\vdots \\
 \mathbf{b}_M &= \parallel 0 \dots 0 \mid 0 \dots 1-\mu \mid 0 \ 0 \dots 0 \mid 0 \ 0 \dots 0 \parallel.
 \end{aligned}$$

We consider each of the columns in the central block as a sum of two terms and, applying the multilinear property of the discriminant to the whole matrix $\mathbf{L} - \mu\mathbf{I}$, end up with a sum of 2^M determinants. Many of them are zeros, since they contain two or more columns proportional to the same vector \mathbf{b} . Actually, there are only $M + 1$ non-zero elements in the expansion. The simplest has the structure $\parallel \mathbf{b}_1 \mid \dots \mid \mathbf{b}_M \parallel$ in the second column block. Since the only non-zero elements in this block are the terms $1 - \mu$ belonging to the main diagonal, the determinant of that part is equal to $(1 - \mu)^M \det \mathbf{N}$, where the matrix \mathbf{N} is defined as follows

$$\begin{vmatrix}
 -\mu & \dots & 0 & f_1^{k_0} + c_1 f_1^Y & f_1^{k_1} & \dots & f_1^{k_{L-1}} & f_1^{y_0} + f_1^Y & f_1^{y_1} & \dots & f_1^{y_{L-1}} \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \dots & -\mu & f_M^{k_0} + c_1 f_M^Y & f_M^{k_1} & \dots & f_M^{k_{L-1}} & f_M^{y_0} + f_M^Y & f_M^{y_1} & \dots & f_M^{y_{L-1}} \\
 K^{x_1} & \dots & K^{x_M} & c_2(f^{k_0}) + c_1(f^Y) + c_1 K^y - \mu & c_2(f^{k_1}) & \dots & c_2(f^{k_{L-1}}) & c_2(f^{y_0}) + (f^Y) + K^y & c_2(f^{y_1}) & \dots & c_2(f^{y_{L-1}}) \\
 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 & 0 & \dots & 0 \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \dots & 0 & 0 & 0 & \dots & -\mu & 0 & 0 & \dots & 0 \\
 0 & \dots & 0 & c_1 & 0 & \dots & 0 & 1 - \mu & 0 & \dots & 0 \\
 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & -\mu & \dots & 0 \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\mu
 \end{vmatrix}.$$

The non-zero elements in this matrix have been computed during the proof of Lemma C.1. Namely, the constant $c_1 = Y^k = -y^*/(1 + g)$, the values of K^{x^m} are given in (C.7), $c_2 = (1 + g)/(x_\diamond^*(1 - x_\diamond^*))$ comes from (C.9), and the derivative $K^y = x_\diamond^*/(1 - x_\diamond^*)$ is computed from (C.2). Finally, by (f^{k_l}) and (f^{y_l}) for $l = 1, \dots, L$, as well as (f^Y) we mean the averages of the corresponding derivatives of the survivors' investment functions weighted by their equilibrium wealth shares.

Coming back to the computation of $\det(\mathbf{L} - \mu\mathbf{I})$, recall that there are other M non-zero blocks in the sum for this determinant. They are obtained when in the second column block all the vectors are \mathbf{b}_i apart from the one column $\nu\mathbf{b}$. But all these determinants can be simplified since $M - 1$ of their columns have only one non-zero element $1 - \mu$ on the diagonal, and after eliminating the corresponding columns and rows the remaining column in the second block will contain the element $-\nu\varphi_v^*$ in the diagonal and zero elements in other positions. Therefore

$$\det(\mathbf{L} - \mu\mathbf{I}) = (1 - \mu)^M \det \mathbf{N} - (1 - \mu)^{M-1} \cdot \frac{g - r_f}{1 + g} \sum_{v=1}^M \varphi_v^* \det \mathbf{N} = (1 - \mu)^{M-1} \left(1 - \mu - \frac{g - r_f}{1 + g} \right) \det \mathbf{N}. \tag{D.1}$$

From this expression we obtain the eigenvalue equal to 1 of multiplicity $M - 1$. Notice that when $M = 1$ there are no such eigenvalues. That is why the system with one survivor is asymptotically stable (of course if all the roots of polynomial (6.2) are inside the unit circle.) When $M > 1$ the eigenvalue 1 obviously corresponds to the movement of the system along the manifold of equilibria. Therefore, it is only the wealth distribution which is changing in the equilibria but not the other quantities.

Another eigenvalue obtained in the expansion (D.1) is $(1 + r_f)/(1 + g)$. It does not affect the stability, since $r_f < g$. All the remaining eigenvalues can be obtained from $(M + 2L) \times (M + 2L)$ matrix \mathbf{N} . We expand this matrix on the minors of the elements of the first row in the last block. Simplifying the resulting minors, we get

$$\det \mathbf{N} = (-1)^L c_1 \mu^{L-1} \det \mathbf{N}_1(M) + (1 - \mu)(-\mu)^{L-1} \det \mathbf{N}_2(M), \tag{D.2}$$

where

$$\mathbf{N}_1(M) = \begin{pmatrix} -\mu & \dots & 0 & f_1^{y0} + f_1^Y & f_1^{y1} & \dots & f_1^{yL-2} & f_1^{yL-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{y0} + f_M^Y & f_M^{y1} & \dots & f_M^{yL-2} & f_M^{yL-1} \\ K^{x1} & \dots & K^{xM} & c_2((f^{y0}) + (f^Y)) + K^y & c_2(f^{y1}) & \dots & c_2(f^{yL-2}) & c_2(f^{yL-1}) \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{pmatrix},$$

and

$$\mathbf{N}_2(M) = \begin{pmatrix} -\mu & \dots & 0 & f_1^{k0} + c_1 f_1^Y & f_1^{k1} & \dots & f_1^{kL-2} & f_1^{kL-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{k0} + c_1 f_M^Y & f_M^{k1} & \dots & f_M^{kL-2} & f_M^{kL-1} \\ K^{x1} & \dots & K^{xM} & c_2((f^{k0}) + c_1(f^Y)) + c_1 K^y - \mu & c_2(f^{k1}) & \dots & c_2(f^{kL-2}) & c_2(f^{kL-1}) \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{pmatrix},$$

The determinants of these two matrices of similar structure are computed in a recursive way. The following lemma is used.

Lemma Appendix D.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & 0 \\ 0 & 0 & 0 & \dots & 1 & -\mu \end{vmatrix} = (-1)^{n+1} \sum_{k=1}^n x_k \mu^{n-k}, \tag{D.3}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k , whose corresponding sign is $(-1)^{k+1}$, is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-k}$ and the relation to be proved immediately follows. □

Consider now the expansion of the matrix $\mathbf{N}_1(M)$ by the minors of the elements from the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\mathbf{N}_1(M)$, which we denote as $\mathbf{N}_1(M - 1)$. The minor associated with K^{x1} has a left upper block with $M - 1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{M-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type (D.3). Applying Lemma D.1 one then has

$$\det \mathbf{N}_1(M) = (-\mu) \det \mathbf{N}_1(M - 1) + (-1)^M K^{x1} \mu^{M-1} (-1)^{L+1} \left(f_1^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_1^{yl} \mu^{L-1-l} \right).$$

Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end the lower right block of the original matrix remains, which is again a matrix similar to (D.3). Applying once more Lemma D.1 one has

for $N_1(M)$ the following

$$\begin{aligned} \det N_1(M) &= (-1)^{M+L+1} \mu^{M-1} \sum_{m=1}^M K^{x_m} \left(f_m^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_m^{y_l} \mu^{L-1-l} \right) \\ &\quad + (-1)^{M+L+1} \mu^M \left((K^Y + c_2 \langle f^Y \rangle) \mu^{L-1} + c_2 \sum_{l=0}^{L-1} \langle f^{y_l} \rangle \mu^{L-1-l} \right) \\ &= (-1)^{M+L+1} \mu^{M-1} \left[\mu^L K^Y + \left(\langle f^Y \rangle \mu^{L-1} + \sum_{l=0}^{L-1} \langle f^{y_l} \rangle \mu^{L-1-l} \right) \cdot \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right]. \end{aligned}$$

The determinant of matrix $N_2(M)$ can be computed analogously and we obtain

$$\begin{aligned} \det N_2(M) &= (-1)^{M+L+1} \mu^{M-1} \sum_{m=1}^M K^{x_m} \left(c_1 f_m^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_m^{k_l} \mu^{L-1-l} \right) \\ &\quad + (-1)^{M+L+1} \mu^M \left(-\mu^L + c_1 (K^Y + c_2 \langle f^Y \rangle) \mu^{L-1} + c_2 \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right) \\ &= (-1)^{M+L+1} \mu^{M-1} \left[-\mu^{L+1} + c_1 \mu^L K^Y + \left(c_1 \langle f^Y \rangle \mu^{L-1} + \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right) \cdot \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right]. \end{aligned}$$

Plugging the two last expressions in (D.2) we finally obtain

$$\begin{aligned} \det N &= (-1)^{M+1} \mu^{M+L-2} c_1 \left[\mu^L K^Y + \left(\langle f^Y \rangle \mu^{L-1} + \sum_{l=0}^{L-1} \langle f^{y_l} \rangle \mu^{L-1-l} \right) \cdot \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right] \\ &\quad + (1-\mu)(-1)^M \mu^{M+L-2} \left[-\mu^{L+1} + c_1 \mu^L K^Y + \left(c_1 \langle f^Y \rangle \mu^{L-1} + \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right) \cdot \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right] \\ &= (-1)^{M+1} \mu^{M+L-2} \left[(1-\mu)\mu^{L+1} + c_1 \mu^L K^Y (1 - (1-\mu)) + \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right. \\ &\quad \times \left. \left(c_1 \langle f^Y \rangle \mu^{L-1} (1 - (1-\mu)) + c_1 \sum_{l=0}^{L-1} \langle f^{y_l} \rangle \mu^{L-1-l} - (1-\mu) \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right) \right] \\ &= (-1)^{M+1} \mu^{M+L-2} (1 + c_1 K^Y - \mu) \\ &\quad \times \left[\mu^{L+1} - c_2 (c_1 \langle f^Y \rangle \mu^L + c_1 \sum_{l=0}^{L-1} \langle f^{y_l} \rangle \mu^{L-1-l} - (1-\mu) \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l}) \right] \end{aligned}$$

where in the last equality we used the relation $-c_2(1 + c_1 K^Y) = -(1 + r_f)/(x_\diamond^*(1 - x_\diamond^*))$, which can be easily checked using the definitions of the constants c_2, c_1 and K^Y .

Thus, we have found another zero eigenvalue of multiplicity $M + L - 2$ and yet another eigenvalue $1 + c_1 K^Y = (1 + r_f)/(1 + g)$ which lies inside the unit circle since $r_f < g$. The stability will depend only on the roots of the polynomial in the squared brackets. After some simplifications and using the relation $x_\diamond^*(1 - x_\diamond^*) = -y^* l'(y^*)$, which can be directly checked from the definition of the Equilibrium Market Curve, we get the polynomial (6.2).

Appendix E. Proof of Proposition 5.5

In Proposition 5.1 we have proved that the system is well defined on \mathcal{D} given in (5.2). Along the same lines it is straightforward to show that T is also well defined on \mathcal{D}' . In particular, an extension for zero dividend yield does not create any problem. Since $\mathcal{D} \subset \mathcal{D}'$, the steady-states defined in Proposition 5.2 are also steady-states in \mathcal{D}' . In those points, of course, $y^* \neq 0$.

In all other steady-states $y^* = 0$, while other quantities are obtained again from (B.1). From the third equation it immediately follows that $k^* = r_f$. Thus, the investment in the risky and the riskless asset yields the same return. Therefore, the wealths of all the agents increase with the same rate, the second equations (B.1) are always satisfied, and no other restrictions on the agents' wealth shares are required.

Appendix F. Proof of Proposition 5.6

The procedure in this proof is analogous to the one we use for proving Propositions 5.3 and 5.4. In particular we use the derivatives and the general Jacobian structure which has been derived in Appendix C. The next Lemma, which is analogous to Lemma C.1 describes the Jacobian matrix for the steady-states with zero yield.

Lemma Appendix F.1. *Let \mathbf{x}^* be a steady-state of dynamics (2.7) described in Proposition 5.5 and let the first M agents survive at this equilibrium. The corresponding Jacobian matrix, $\mathbf{J}(\mathbf{x}^*)$, has the following structure, where the actual values of non-zero elements, denoted by the symbols $*$, are varying.*

$$\begin{array}{cccccccccccccccc}
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \left[\frac{\partial \mathcal{X}}{\partial \mathcal{K}} \right] & & & & & \left[\frac{\partial \mathcal{X}}{\partial \mathcal{Y}} \right] & & & \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\
 * & \dots & * & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\
 \left[\Phi^x \right] & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \left[\Phi^k \right] & & & & & \left[\Phi^y \right] & & & \\
 * & \dots & * & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 * & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & * & * & * & \dots & * & * \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \frac{1+g}{1+r_f} & 0 & \dots & 0 & 0 \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
 \end{array}$$

The solid lines divide the matrix in the same 16 blocks as in (C.4). In addition, the first and second column-blocks and second row-block are split into two parts of sizes M and $N - M$, corresponding to the survivors and the non-survivors, respectively.

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial \mathcal{X} / \partial \mathcal{X}$ and $\partial \mathcal{X} / \partial \mathcal{W}$, are always zero. Two other blocks, $\partial \mathcal{X} / \partial \mathcal{K}$ and $\partial \mathcal{X} / \partial \mathcal{Y}$, in general contain non-zero elements and simplified, because in the equilibrium $k^* = r_f$ and $y^* = 0$, and therefore $Y^y = (1 + g) / (1 + r_f)$ and $Y^k = 0$.

To simplify the remaining row-blocks, notice from (C.3) that $\Phi_n^{x^m} = 0$ and $\Phi_n^{\varphi^m} = \delta_n^m$, while $\Phi_n^k = \Phi_n^y = \varphi_n^*(x_n^* - \langle x \rangle) / (1 + r_f)$ at this equilibrium. This follows immediately from the relation $k^* + y^* - r_f = 0$. At the same time from (C.2) we have $K^{\varphi^m} = 0$, $K^y = \langle x^{*2} \rangle / \langle x^*(1 - x^*) \rangle$, and $K^{z^m} = -K^{x^m} = \varphi_m^*(1 + r_f) / \langle x^*(1 - x^*) \rangle$.

Thus, in the first block of the second row-block, $\partial \mathcal{W} / \partial \mathcal{X}$, the elements are equal to $\Phi_n^k \cdot K^{x^m}$ and they are zeros as soon as either n or m is larger than M . In the next block, $\partial \mathcal{W} / \partial \mathcal{W}$, all the elements are zeros, apart from the main diagonal elements. All the elements of the next block, $\partial \mathcal{W} / \partial \mathcal{K}$, contain the multiplying term Φ_n^k , so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^k]$. Similarly, in the block $\partial \mathcal{W} / \partial \mathcal{Y}$ all the elements are the sums containing either the term Φ_n^k or the term Φ_n^y , so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^y]$.

In the next row-block, with L rows, the elements are zeros in all the lines but the first. The only exception from this rule are the elements below the main diagonal in the block $\partial \mathcal{K} / \partial \mathcal{K}$ which are all equal to 1. For the elements in the first row we use the derivatives of function K derived above. Consequently, in the first block, $\partial \mathcal{K} / \partial \mathcal{X}$, for the non-surviving agents we have $K^{x^m} = 0$. Analogously, in the next block, $\partial \mathcal{K} / \partial \mathcal{W}$, all the elements are zeros. The simplifications in the blocks $\partial \mathcal{K} / \partial \mathcal{K}$ and $\partial \mathcal{K} / \partial \mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{z^m} for all the non-survivors are zeros, therefore all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. \square

In the remaining part of this proof we identify different multipliers of the matrix derived in the previous Lemma. From the first line in the fourth row-block we immediately obtain the eigenvalue $(1 + g) / (1 + r_f)$ and condition $g < r_f$ for stability.

Elimination of this line together with the corresponding column creates a zero line in the same block. Proceeding recursively, we obtain the eigenvalue 0 with multiplicity $L - 1$ and eliminate the fourth line- and column-block entirely.

From the second column-block we get the eigenvalue 1 with multiplicity N . These eigenvalues correspond to the directions of change in the wealth distribution between different agents. (Recall from Proposition 5.5 that the wealth shares are free of choice.) Consequently, there are no asymptotically stable equilibria. At the same time, it is clear that these eigenvalues, lying on the border of the unit circle, do not prevent the steady-state from stability.

From the last $N - M$ columns of the first column-block we obtain the eigenvalue 0 with multiplicity $N - M$. Eliminating the corresponding columns and rows we get the following matrix

$$N_3(M) = \begin{pmatrix} -\mu & \dots & 0 & f_1^{k_0} & f_1^{k_1} & \dots & f_1^{k_{L-2}} & f_1^{k_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{k_0} & f_M^{k_1} & \dots & f_M^{k_{L-2}} & f_M^{k_{L-1}} \\ K^{x_1} & \dots & K^{x_M} & \sum_m K^{z_m} f_m^{k_0} - \mu & \sum_m K^{z_m} f_m^{k_1} & \dots & \sum_m K^{z_m} f_m^{k_{L-2}} & \sum_m K^{z_m} f_m^{k_{L-1}} \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{pmatrix},$$

where, as we found in Lemma F.1, the derivatives are $K^{z_m} = -K^{x_m} = \varphi_m^*(1 + r_f) / (x^*(1 - x^*))$. This matrix has the same functional form as matrices $N_1(M)$ and $N_2(M)$ whose discriminant we computed in Appendix D. Proceeding in analogous way, we get

$$\begin{aligned} \det N_3(M) &= (-1)^{M+L+1} \mu^{M-1} \sum_{m=1}^M K^{x_m} \left(\sum_{l=0}^{L-1} f_m^{k_l} \mu^{L-1-l} \right) \\ &\quad + (-1)^{M+L+1} \mu^M \left(-\mu^L + \sum_{l=0}^{L-1} \sum_m K^{z_m} f_m^{k_l} \mu^{L-1-l} \right) \\ &= -(-1)^{M+L+1} \mu^{M-1} \frac{1 + r_f}{\langle x^*(1 - x^*) \rangle} \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \\ &\quad + (-1)^{M+L+1} \mu^M \left(-\mu^L + \frac{1 + r_f}{\langle x^*(1 - x^*) \rangle} \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right) \\ &= (-1)^{M+L} \mu^{M-1} \left(\mu^{L+1} + \frac{1 + r_f}{\langle x^*(1 - x^*) \rangle} (1 - \mu) \sum_{l=0}^{L-1} \langle f^{k_l} \rangle \mu^{L-1-l} \right). \end{aligned}$$

Appendix G. Proof of Corollary 6.1 and Proposition 6.1

When $L = 1$ the polynomial $\tilde{Q}(\mu)$ in (6.2) can be simplified and it is given by

$$\mu^2 + \mu C \frac{1+g}{y^*} - C \left(1 + \frac{1+g}{y^*} \right),$$

where $C = \sum_{m=1}^M f'_m(y^* + g) \varphi_m^* / l(y^*)$. Let us introduce two quantities, trace and determinant, as follows $\text{Tr} = -C(1 + g)/y^*$ and $\text{Det} = -C(1 + (1 + g)/y^*)$. According to standard results for the second-degree polynomial (see e.g. Medio and Lines (2001)), we get the following conditions for stability, whose equality correspond to the bifurcation loci of fold, flip and Neimark-Sacker bifurcation respectively,

$$1 - \text{Tr} + \text{Det} > 0, \quad 1 + \text{Tr} + \text{Det} > 0, \quad \text{and} \quad \text{Det} < 1.$$

Using our definitions of Tr and Det we get the conditions $C < 1$, $C < y^*/(y^* + 2(1 + g))$ and $C > -y^*/(1 + g + y^*)$ respectively. The first condition is redundant, while the last two give (6.3).

For larger L the results on stability are limited. First, we can derive the loci of fold and flip bifurcations substituting, respectively, $\mu = 1$ and $\mu = -1$ into polynomial $\tilde{Q}(\mu)$ in (6.2). Straight-forward computations show that the line $C = 1$ is a locus of fold bifurcation for any L , while the curve $C = y^*/(y^* + 2(1 + g))$ is a locus of flip bifurcation for any odd L (and there is no flip bifurcation, when L is even).

Second, plugging $\mu = e^{i\psi}$, where ψ is arbitrary angle and i is the imaginary unit, into equation $\tilde{Q}(\mu) = 0$ we can derive the locus of Neimark-Sacker bifurcation. In case of $L = 2$ the equation can be solved and, after tedious computations, one get the condition

$$C^2 (y^{*2} + 3(1 + g)y^* + 2(1 + g)^2) + 2C y^{*2} - 4y^{*2} = 0.$$

This second-order curve is depicted in the right panel of Fig. 7 in coordinates (y^*, C) .

Finally, we analyze the case $L \rightarrow \infty$. Rewrite polynomial (6.2) as follows

$$\tilde{Q}(\mu) = \mu^{L-1} \left(\mu^2 - \frac{1}{L} \frac{1 - (1/\mu)^L}{1 - 1/\mu} \left(1 + (1 - \mu) \frac{1+g}{y^*} \right) C \right). \quad (\text{G.1})$$

We want to prove that all the roots of this polynomial lie inside the unit circle of the complex plane for L high enough. Consider the region outside the unit circle (including the circle itself), fix $\mu = \mu_0$ and let $L \rightarrow \infty$. Since $|\mu_0| \geq 1$, the first term in (G.1) cannot be equal to zero. Therefore, μ_0 can be a root of the characteristic polynomial only if the expression in the parenthesis cancels out. First, assume that $|\mu_0| > 1$. Then when $L \rightarrow \infty$ the expression in the parenthesis leads to $\mu_0 = 0$ which contradicts our choice of μ_0 . Second, let $|\mu_0| = 1$ but $\mu_0 \neq 1$. In this case the expression $|1 - (1/\mu)^L|$ is bounded (uniformly with L), and so, again taking the limit $L \rightarrow \infty$ we obtain $\mu_0 = 0$. Thus, the only remaining possibility is $\mu_0 = 1$, which implies $C = 1$, that is, the locus of fold bifurcations. Since we know that when the relative slope is $C = 0$, the steady-state is stable by continuity it follows that whenever $C < 1$ the steady state is stable too. This implies (6.4), and proofs Corollary 6.1.

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